Optimized Democracy (Spring 2023)
Problem Set #3

Due: 3/10/2023 11:59pm ET

Instructions:

- It is fine to look up a complicated sum or inequality, but please do not look up an entire solution. In particular, the solutions to many of the problems that we give can be found in papers, but, needless to say, you should avoid reading the proof if you come across the relevant paper. If for some reason you did see the solution before working it out yourself, please say so in your solution.
- You may discuss the problems with classmates but please write down solutions completely on your own.
- Please *type up* your solution and submit to Gradescope.

Problems:

1. In class we discussed notions of proportionality for approval-based elections like extended justified representation (EJR), which only guarantees that one voter in each "deserving" coalition is satisfied. In this problem our goal is to provide guarantees that hold *on average*.

   Recall that in the approval-based committee elections settings we have a set N of n voters and a target committee size k, where each voter $i \in N$ approves a set of alternatives $\alpha_i \subseteq A$. Let $q := \frac{n}{k}$. We say that a set of $S \subseteq N$ of voters is *ℓ-cohesive* if $|S| \geq \ell \cdot q$ and $|\bigcap_{i \in S} \alpha_i| \geq \ell$. As in class, we write $u_i(W) = |W \cap \alpha_i|$.

   (a) **[15 points]** Assume that q is an integer. Suppose that a committee $W \subseteq A$, $|W| = k$, satisfies Extended Justified Representation (EJR), so for each $1 \leq \ell \leq k$ and every ℓ-cohesive group S, there exists $i \in S$ with $u_i(W) \geq \ell$. Now let S be an ℓ-cohesive group with $|S| = \ell \cdot q$. Prove that

   $$\sum_{i \in S} \frac{1}{|S|} u_i(W) \geq \frac{\ell - 1}{2},$$

   that is, S obtains high average utility.

   Note: With more work it can be shown that it is possible to achieve average utility at least $\ell - 1$ for ℓ-cohesive groups.

(b) **[25 pt]** Prove that for all $\varepsilon > 0$, there exists an election such that, no matter which committee is chosen, there is a 1-cohesive group S that has average utility at most ε. More formally, there is a set N of n voters, a set A of m alternatives, target committee size k, and approval set $\alpha_i \subseteq A$ for each $i \in N$, such that the following holds. For all committees $W \subseteq A$ with $|W| = k$, there is a set of voters S with $|S| \geq n/k$ and $|\bigcap_{i \in S} \alpha_i| \geq 1$ such that,

$$\sum_{i \in S} \frac{1}{|S|} u_i(W) \leq \varepsilon.$$

Note: With more work it can be shown that there exist elections such that no matter which committee is chosen, there is an ℓ-cohesive group with average utility at most $\ell - 1 + \varepsilon$. This means the lower bound mentioned in the note in part (a) is tight.

Hint: Construct a family of instances with $m = k + 1$ alternatives where each voter approves either one or two alternatives.

2. Consider the cake cutting problem with n players and valuation functions $V_1, \ldots, V_n$ satisfying additivity, normalization, and divisibility. Denote the *social welfare* of an allocation $\boldsymbol{A}$ by $\mathrm{sw}(\boldsymbol{A}) = \sum_{i=1}^{n} V_i(A_i)$.[1]

   We are interested in the "price" of proportionality, that is, we wish to quantify the worst-case loss in social welfare due to the proportionality constraint. The problems below give asymptotically tight upper and lower bounds on this "price."

   (a) **[35 points]** Show that, for all valuation functions $V_1, \ldots, V_n$ and any allocation $\boldsymbol{A}^\star$, there exists a *proportional* allocation $\boldsymbol{A}$ such that $\mathrm{sw}(\boldsymbol{A}^\star)/\mathrm{sw}(\boldsymbol{A}) = O(\sqrt{n})$.

   Hint: For an arbitrary allocation $\boldsymbol{A}^\star$, let $L = \{i \in N : V_i(A_i^\star) \geq 1/\sqrt{n}\}$. Analyze two cases: $|L| \geq \sqrt{n}$ and $|L| < \sqrt{n}$. The latter case is easy. For the former case, the idea is to convert $\boldsymbol{A}^\star$ into a proportional allocation with high social welfare, as follows. For each $i \in N \setminus L$, reallocate $A_i^\star$ among players in $N \setminus L$ using a proportional allocation; and for each $i \in L$, reallocate $A_i^\star$ among the players in $\{i\} \cup (N \setminus L)$ using a proportional allocation with $\sqrt{n}$ "copies" of player i.

   (b) **[25 points]** Give a family of examples of $V_1, \ldots, V_n$ (one example for each value of n) such that, for each example, there exists an allocation $\boldsymbol{A}^\star$ such that for any *proportional* allocation $\boldsymbol{A}$, $\mathrm{sw}(\boldsymbol{A}^\star)/\mathrm{sw}(\boldsymbol{A}) = \Omega(\sqrt{n})$.

[1] Since we are talking about social welfare, the normalization assumption is no longer without loss of generality.