Optimized Democracy

Spring 2022 | Lecture 6

The Epistemic Approach

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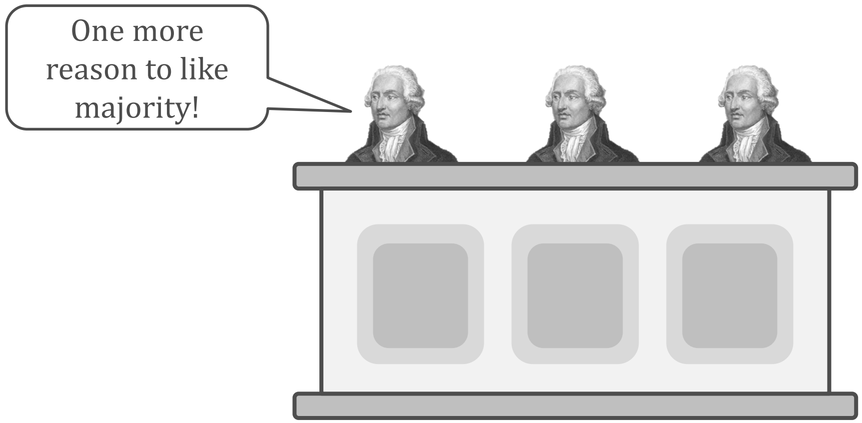
CONDORCET STRIKES AGAIN

- For Condorcet, the purpose of voting is not merely to balance subjective opinions; it is a collective quest for the truth
- Enlightened voters try to judge which alternative best serves society
- This is an arguable model of political elections, but there are certainly settings where the ground-truth assumption holds true

CONDORCET JURY THEOREM



Theorem [Condorcet 1785]: Suppose that there is a correct alternative and an incorrect alternative, and there are n voters, each of whom votes independently for the correct alternative with probability $p > 1/2$, then the probability that the majority would be correct goes to 1 as $n \to \infty$

CONDORCET JURY THEOREM

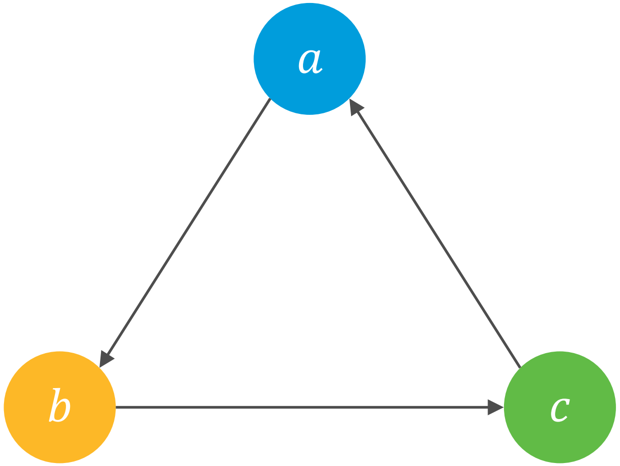
- The (modern) proof follows directly from the (weak) law of large numbers
- Lemma: Let $X_1, X_2, \ldots$ be an infinite sequence of i.i.d. random variables with expectation μ, then for any $\epsilon > 0$,
  $$\lim_{n\to\infty} \Pr\left[|\bar{X}_n - \mu| < \epsilon\right] = 1$$
- Now take $\epsilon = p - 1/2$

THE CASE OF $m \geq 3$

- In Condorcet's general model there is a true ranking of the alternatives
- Each voter evaluates every pair of alternatives independently, gets the comparison right with probability $p > 1/2$
- The results are tallied in a voting matrix
- Condorcet's proposal: Find the "most probable" ranking by taking the majority opinion for each comparison; if a cycle forms, "successively delete the comparisons that have the least plurality"

CONDORCET'S "SOLUTION"

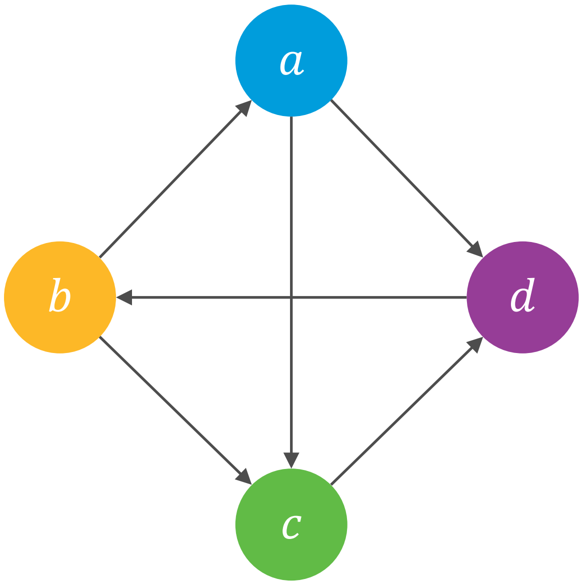
	a	b	c
a	-	8	6
b	5	-	11
c	7	2	-



Delete $c \succ a$ to get $a \succ b \succ c$

CONDORCET'S "SOLUTION"

	a	b	c	d
a	-	12	15	17
b	13	-	16	11
c	10	9	-	18
d	8	14	7	-



Order of strength is $c \succ d$, $a \succ d$, $b \succ c$, $a \succ c$, $d \succ b$, $b \succ a$; deleting $b \succ a$ leaves a cycle; deleting $d \succ b$ creates ambiguity

CONDORCET'S "SOLUTION"

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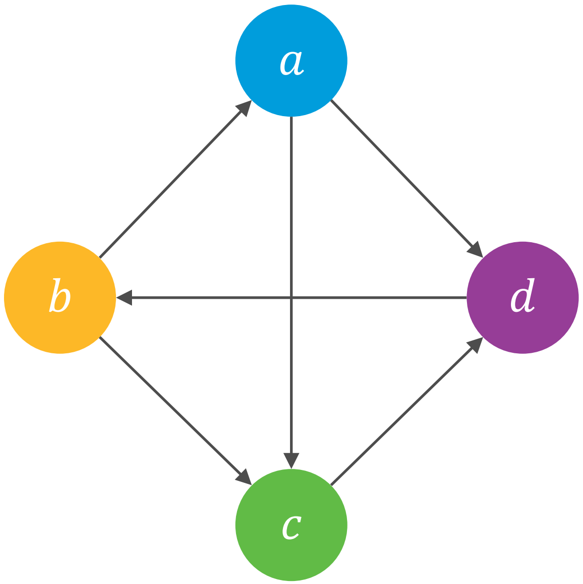


Did Condorcet mean we should reverse the weakest comparisons? If we reverse $b \succ a$ and $d \succ b$, we get $a \succ b \succ c \succ d$, with 89 votes, but reversing $d \succ b$ leads to $b \succ a \succ c \succ d$ with 90 votes

Isaac Todhunter

1820–1884

“The obscurity and self-contradiction are without any parallel, so far as our experience of mathematical works extends ... no amount of examples can convey an adequate impression of the evils.”

YOUNG'S SOLUTION

- M is the matrix of votes and π is the true ranking
- MLE maximizes $\Pr[M \mid \pi]$
- Suppose true ranking is $a \succ_\pi b \succ_\pi c$; prob. of observations $\Pr[M \mid \pi]$:

$$\binom{13}{8} p^8(1-p)^5 \cdot \binom{13}{6} p^6(1-p)^7 \cdot \binom{13}{11} p^{11}(1-p)^2$$

- For $a \succ_\pi c \succ_\pi b$, $\Pr[M \mid \pi]$ is

$$\binom{13}{8} p^8(1-p)^5 \cdot \binom{13}{6} p^6(1-p)^7 \cdot \binom{13}{2} p^2(1-p)^{11}$$

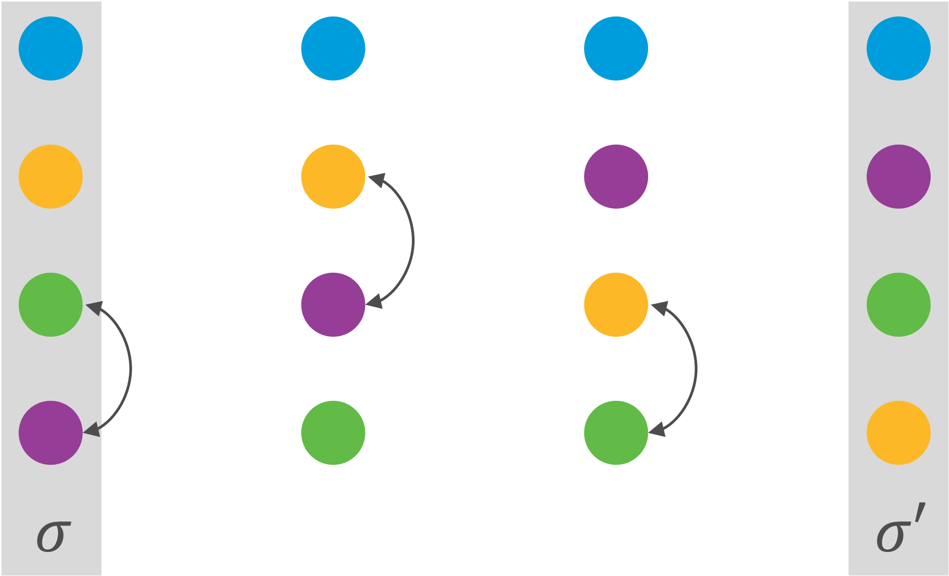
- Binomial coefficients are identical, so

$$\Pr[M \mid \pi] \propto p^{\#agree}(1-p)^{\#disagree}$$

	a	b	c
a	-	8	6
b	5	-	11
c	7	2	-

THE KENDALL TAU DISTANCE

- The Kendall tau distance between σ and σ' is defined as
$$d_{KT}(\sigma,\sigma') = \left|\left\{\{a,b\}\colon\ a \succ_\sigma b \wedge b \succ_{\sigma'} a\right\}\right|$$
- Can be thought of as "bubble sort distance"



THE MALLOWS MODEL

- Defined by parameter $\phi \in (0,1]$
- Probability of a voter having the ranking σ given true ranking π is

$$\Pr[\sigma|\pi] = \frac{\phi^{d_{KT}(\sigma,\pi)}}{\sum_{\tau} \phi^{d_{KT}(\tau,\pi)}}$$

- Same as the Condorcet noise model where the process “restarts” if a cycle forms and

$$\phi = \frac{1-p}{p}$$

THE KEMENY RULE

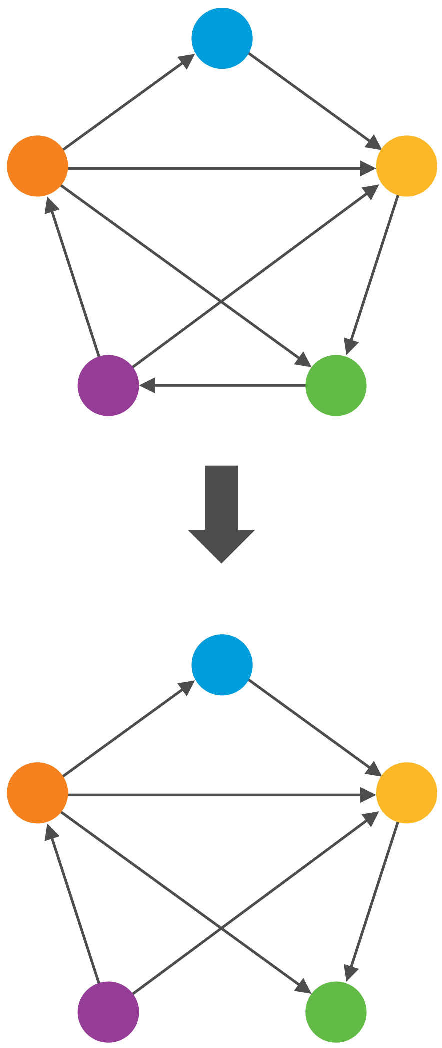
- What is probability of observing profile $\boldsymbol{\sigma}$ given true ranking π?
- Denote $Z_\phi = \sum_\tau \phi^{d_{KT}(\tau,\pi)}$, then

$$\Pr[\boldsymbol{\sigma} \mid \pi] = \prod_{i \in N} \frac{\phi^{d_{KT}(\sigma_i,\pi)}}{Z_\phi} = \frac{\phi^{\sum_{i \in N} d_{KT}(\sigma_i,\pi)}}{\left(Z_\phi\right)^n}$$

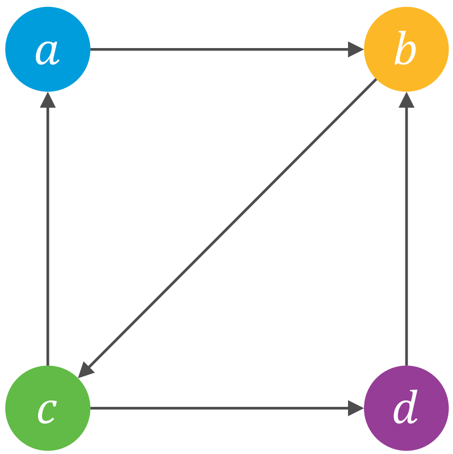
- The MLE is clearly the Kemeny Rule: Given a preference profile $\boldsymbol{\sigma}$, return a ranking π that minimizes $\sum_{i \in N} d_{KT}(\sigma_i, \pi)$

COMPLEXITY OF KEMENY

- Theorem: Computing the output of the Kemeny rule is NP-hard
- The proof exploits a connection to the Minimum Feedback Arc Set Problem: Given a directed graph $G = (V, E)$ and $L \in \mathbb{N}$, is there $F \subseteq E$ s.t. $|F| \leq L$ and $(V, E \setminus F)$ is acyclic?



PROOF IDEA



(a,b)		(b,c)		(c,a)		(c,d)		(d,b)	
a	d	b	d	c	d	c	b	d	c
b	c	c	a	a	b	d	a	b	a
c	a	a	b	b	c	a	c	a	d
d	b	d	c	d	a	b	d	c	b

For each edge create a pair of voters that agree on the corresponding ordered pair of alternatives and disagree on everything else; there's an acyclic subgraph that deletes k edges if and only if there is a ranking that (beyond the inevitable disagreements) disagrees with k pairs of voters

KEMENY IN PRACTICE

In practice Kemeny computation is typically formulated as an integer linear program: For every $a, b \in A$, $x_{(a,b)} = 1$ iff a is ranked above b, and $w_{(a,b)} = \left|\{i \in N: a \succ_{\sigma_i} b\}\right|$

minimize $\sum_{(a,b)} x_{(a,b)} w_{(b,a)}$
subject to:
for all distinct $a, b \in A$, $x_{(a,b)} + x_{(b,a)} = 1$
for all distinct $a, b, c \in A$, $x_{(a,b)} + x_{(b,c)} + x_{(c,a)} \leq 2$
for all distinct $a, b \in A$, $x_{(a,b)} \in \{0,1\}$

AN AXIOMATIC VIEWPOINT

The axiomatic viewpoint isn't necessarily at odds with the epistemic viewpoint; how does Kemeny fare when examined through an axiomatic lens?

Poll

Which of the following axioms is satisfied by Kemeny?

- Condorcet consistency
- Unanimity
- Both axioms
- Neither one

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