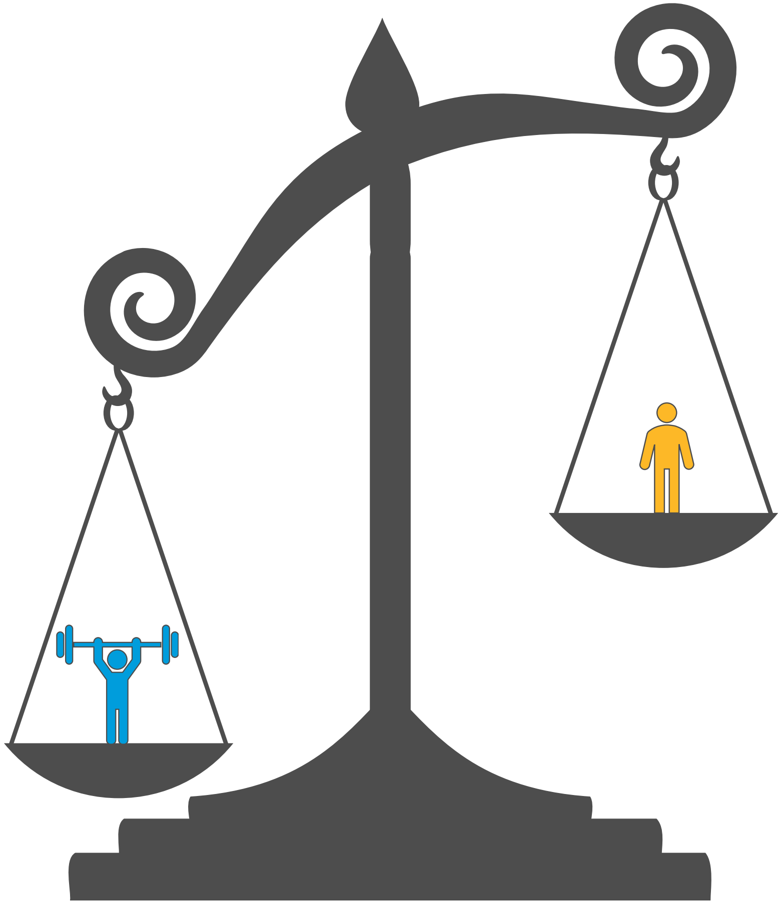
Optimized Democracy

Spring 2023 | Lecture 19

The Electoral College

Ariel Procaccia | Harvard University

WEIGHTED VOTING

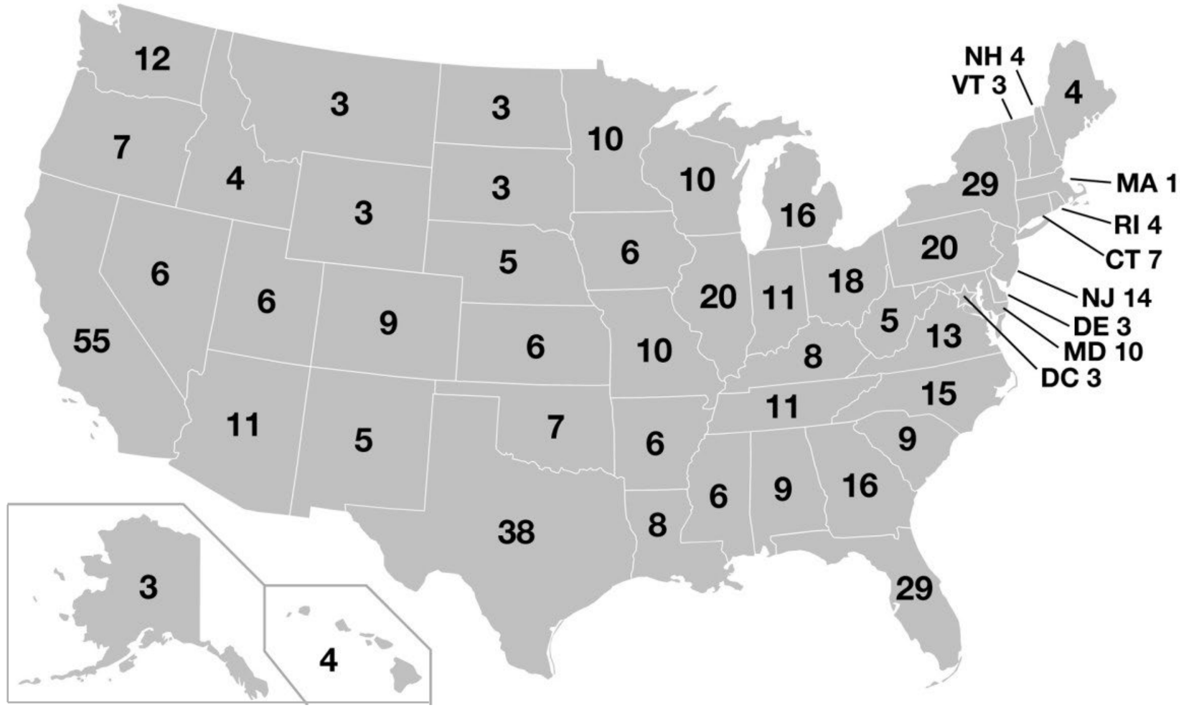


How does weighted voting distribute power?

ORIGINS OF THE ELECTORAL COLLEGE*

*Electoral College not shown in picture

WEIGHTED VOTING IN THE EC



Each of 50 states and DC is seen as a weighted voter with weight equal to the state's members of Congress (DC has a weight of 3). This assumes that the electors of each state vote as a bloc and ignores the special rules of Maine and Nebraska.

WEIGHTED VOTING GAMES

- A simple cooperative game is a pair (N, v) where $N = \{1, \ldots, n\}$ is the set of players and $v: 2^N \rightarrow \{0,1\}$ is the value function; assume that if $v(S) = 1$ and $S \subseteq T$ then $v(T) = 1$
- If $v(S) = 1$ we say S is a winning coalition, otherwise it is a losing coalition
- A weighted voting game is the simple cooperative game defined by a quota q and weight $w_i \in \mathbb{N} \cup \{0\}$ for all $i \in N$, where for all $S \subseteq N$, $v(S) = 1$ if and only if $\sum_{i \in S} w_i \geq q$

Question

Consider the simple cooperative game with $n = 4$ where the winning coalitions are $\{1,2\}$, $\{3,4\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, $\{1,2,3,4\}$. Can this game be represented as a weighted voting game?

DUMMY PLAYERS

- A dummy player is $i \in N$ such that for all $S \subseteq N \setminus \{i\}$, $v(S) = v(S \cup \{i\})$

Poll 1

Which of the following conditions is a sufficient condition for the existence of a dummy player in a weighted voting game? Which of these conditions is necessary?

- $\exists i \in N$ s.t. $w_i = 0$
- $\exists i \in N$ s.t. $w_i \geq q$ and $\sum_{j \neq i} w_j < q$

EXAMPLE: THE COMMON MARKET

- The common market was formed in 1958 as a federation of six European countries: France, (West) Germany, Italy, Belgium, the Netherlands, and Luxembourg
- It was governed by the council of ministers, which induces a weighted voting game with $q = 12$ and the following weights:

France	Germany	Italy	Belgium	Netherlands	Luxembourg
4	4	4	2	2	1

- Luxembourg is a dummy player!

UN SECURITY COUNCIL

Permanent members | Asia | Western Europe
Africa | Latin America | Eastern Europe

A measure passes if 9 out of 15 members of the Security Council vote in favor, provided that no permanent member votes against it

This is a weighted voting game with $q = 49$, $w_i = 9$ for each permanent member, and $w_i = 1$ for each non-permanent member

VOTING POWER

- In weighted voting, the justification for weights is that voters are entitled to different degrees of influence over the outcome
- Influence means that an alternative would have lost without the support of a voter and wins with it
- How can we measure a voter's overall influence?

John Banzhaf

1940–

Public interest lawyer, activist and professor at George Washington University Law School. Also known for TV advertising.

BANZHAF POWER INDEX

- Given a simple cooperative game (N, v), the Banzhaf power index of player i is
$$\beta_i := \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} v(S \cup \{i\}) - v(S)$$
- It is the probability that player i is pivotal in a uniformly random coalition that contains them

Poll 2

Suppose that in a weighted voting game, $w_i > w_j$. Which of the following situations is possible?

- $\beta_i < \beta_j$
- $\beta_i = 0$
- Both
- Neither one

EXAMPLE: THE COMMON MARKET

- Recall that the players are F, G, I, B, N, L with weights 4, 4, 4, 2, 2, 1 and quota 12
- The table on the right lists all 14 winning coalitions
- Banzhaf power indices are 10/32 for F, G, I; 6/32 for B and N; and 0 for L
- France might argue that it's less represented in comparison to Belgium than its weight would suggest

Coalition	Weight	Pivotal members
FGIBNL	17	None
FGIBN	16	None
FGIBL	15	F, G, I
FGINL	15	F, G, I
FGIB	14	F, G, I
FGIN	14	F, G, I
FGIL	13	F, G, I
FGBNL	13	F, G, B, N
FIBNL	13	F, I, B, N
GIBNL	13	G, I, B, N
FGI	12	F, G, I
FGBN	12	F, G, B, N
FIBN	12	F, I, B, N
GIBN	12	G, I, B, N

COMPLEXITY OF BANZHAF

- **Theorem:** Given a weighted voting game and a player $i \in N$, the problem of computing β_i is #P-complete
- However, the Banzhaf power index is very easy to approximate using a Monte Carlo Algorithm that samples coalitions from the uniform distribution over $2^{N \setminus \{i\}}$ and returns the fraction of coalitions S such that i is pivotal in $S \cup \{i\}$
- **Theorem:** Given $\epsilon, \delta > 0$ and $O\left(\ln\frac{1}{\delta} \cdot \frac{1}{\epsilon^2}\right)$ samples, the above algorithm returns an estimate $\hat{\beta}_i$ such $\left|\beta_i - \hat{\beta}_i\right| < \epsilon$ with probability at least $1 - \delta$

PROOF OF THEOREM

- Lemma (Hoeffding): Let $X_1, \ldots, X_k$ be i.i.d. Bernoulli random variables with $\mathbb{E}\left[X_j\right] = \mu$, then $\Pr\left[\left|\frac{1}{k}\sum_{j=1}^{k} X_j - \mu\right| \geq \epsilon\right] \leq 2\exp(-2k\epsilon^2)$
- For each sample X_j we have that $\mu = \beta_i$
- Plugging in $k = \ln\frac{2}{\delta} \cdot \frac{1}{2\epsilon^2}$ and $\hat{\beta}_i = \frac{1}{k}\sum_{j=1}^{k} X_j$ we get that the probability that $\left|\beta_i - \hat{\beta}_i\right| \geq \epsilon$ is at most $2\exp(-\ln 2/\delta) = \delta$ ■

BANZHAF POWER IN THE ELECTORAL COLLEGE

State	EVs (2000)	Banzhaf
California	55	23.75%
Texas	34	13.31%
New York	31	12.07%
Florida	27	10.43%
Illinois	21	8.05%
Pennsylvania	21	8.05%
Ohio	20	7.66%
Michigan	17	6.49%
New Jersey	15	5.71%
Georgia	15	5.71%
North Carolina	15	5.71%
Virginia	13	4.95%
Massachusetts	12	4.56%
Indiana	11	4.18%
Washington	11	4.18%
Tennessee	11	4.18%
Missouri	11	4.18%
Wisconsin	10	3.79%
Maryland	10	3.79%
Arizona	10	3.79%
Minnesota	10	3.79%
Louisiana	9	3.41%
Alabama	9	3.41%
Colorado	9	3.41%
Kentucky	8	3.03%
South Carolina	8	3.03%
Oklahoma	7	2.65%
Oregon	7	2.65%
Connecticut	7	2.65%
Iowa	7	2.65%
Mississippi	6	2.27%
Kansas	6	2.27%
Arkansas	6	2.27%
Utah	5	1.89%
Nevada	5	1.89%
New Mexico	5	1.89%
West Virginia	5	1.89%
Nebraska	5	1.89%
Idaho	4	1.51%
Maine	4	1.51%
New Hampshire	4	1.51%
Hawaii	4	1.51%
Rhode Island	4	1.51%
Montana	3	1.14%
Delaware	3	1.14%
South Dakota	3	1.14%
North Dakota	3	1.14%
Alaska	3	1.14%
Vermont	3	1.14%
Wyoming	3	1.14%
DC	3	1.14%

WEIGHT VS. BANZHAF

- Let's compare California and North Dakota:
  - The ratio of electoral votes (2000 census) is $55/3 = 18.33$
  - The ratio of Banzhaf power indices is
    $$23.75/1.13 = 21.02$$
  - The ratio of populations (2000 census) is $33{,}093{,}798/756{,}874 = 43.72$
- This demonstrates the "+2 effect" of adding two senators to electoral votes
- Which of these ratios should we focus on?

THE INFLUENCE OF VOTERS

- The Banzhaf power index quantifies the influence of states, but arguably what matters more is the influence of voters
- Assume that each voter independently votes for each of two alternatives with probability 0.5
- Denote by α the probability that a voter casts a tie-breaking vote in their own state that has population $p = 2k + 1$ and Banzhaf power index β
- The probability that our voter affects the outcome of the election is $\alpha \cdot \beta$ (why?)
- We've estimated β so we just need to estimate α

THE INFLUENCE OF VOTERS

- It holds that $\alpha = \frac{\binom{2k}{k}}{2^{2k}}$
- Using Stirling's approximation of factorial and a bit of algebra,

$$\alpha \approx \frac{\frac{2^{2k}}{\sqrt{\pi k}}}{2^{2k}} = \frac{1}{\sqrt{\pi k}} = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{p-1}}$$

- So what is the ratio of $\alpha \cdot \beta$ for voters in different state?

RATIO OF VOTER INFLUENCE ACROSS STATES

State	EVs (2000)	Index
California	55	3.34
Texas	34	2.33
New York	31	2.37
Florida	27	2.06
Illinois	21	1.94
Pennsylvania	21	1.97
Ohio	20	1.95
Michigan	17	1.74
New Jersey	15	1.65
Georgia	15	1.60
North Carolina	15	1.62
Virginia	13	1.52
Massachusetts	12	1.53
Indiana	11	1.43
Washington	11	1.42
Tennessee	11	1.45
Missouri	11	1.49
Wisconsin	10	1.38
Maryland	10	1.35
Arizona	10	1.28
Minnesota	10	1.41
Louisiana	9	1.38
Alabama	9	1.38
Colorado	9	1.35
Kentucky	8	1.27

State	EVs (2000)	Index
South Carolina	8	1.25
Oklahoma	7	1.21
Oregon	7	1.18
Connecticut	7	1.22
Iowa	7	1.33
Mississippi	6	1.14
Kansas	6	1.18
Arkansas	6	1.16
Utah	5	1.02
Nevada	5	1.00
New Mexico	5	1.17
West Virginia	5	1.22
Nebraska	5	1.23
Idaho	4	1.07
Maine	4	1.13
New Hampshire	4	1.12
Hawaii	4	1.13
Rhode Island	4	1.24
Montana	3	1.00
Delaware	3	1.05
South Dakota	3	1.11
North Dakota	3	1.23
Alaska	3	1.18
Vermont	3	1.22
Wyoming	3	1.37

DISCUSSION

- The Electoral College is typically seen as giving an advantage to small states (due to the +2 phenomenon)
- The irony is that, according to the foregoing analysis, the residents of small states have significantly less power than those of large states

BIBLIOGRAPHY

G. E. A. Robinson and D. H. Ullman. A Mathematical Look at Politics. CRC Press, 2010.

Y. K. Prasad and J. S. Kelly. NP-Completeness of Some Problems Concerning Voting Games. International Journal of Game Theory, 1990.

Y. Bachrach, E. Markakis, E. Resnick, A. D. Procaccia, J. S. Rosenschein, and A. Saberi. Approximating Power Indices. Autonomous Agents and Multi-Agent Systems, 2010.