Optimized Democracy

Spring 2023 | Lecture 12

Indivisible Goods

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DIVIDE: RENT FARE CREDIT GOODS TASKS | ABOUT FEEDBACK

PROVABLY FAIR SOLUTIONS.

Spliddit offers quick, free solutions to everyday fair division problems, using methods that provide indisputable fairness guarantees and build on decades of research in economics, mathematics, and computer science.

Share Rent

Split Fare

Assign Credit

Divide Goods

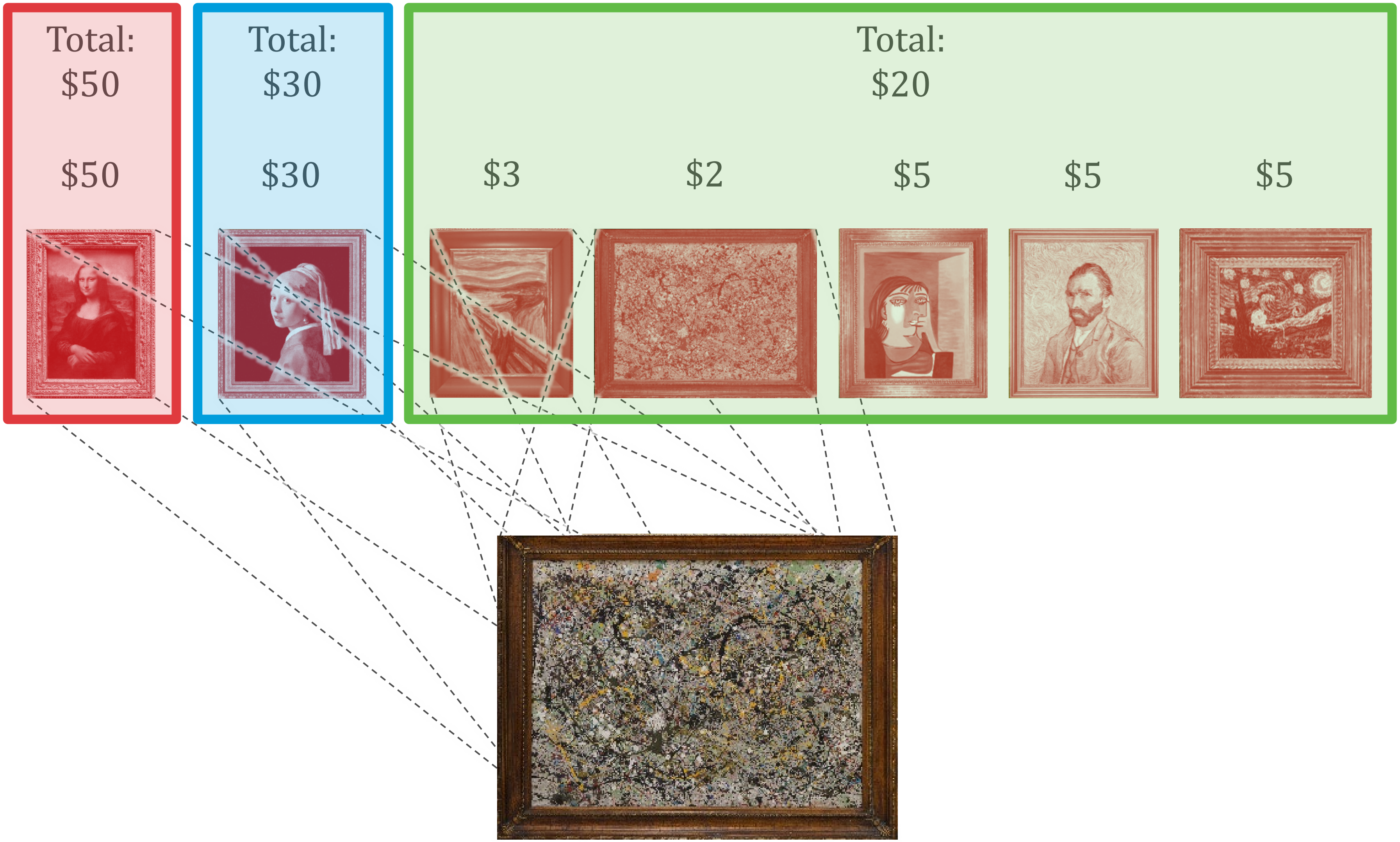
Distribute Tasks

Suggest an App

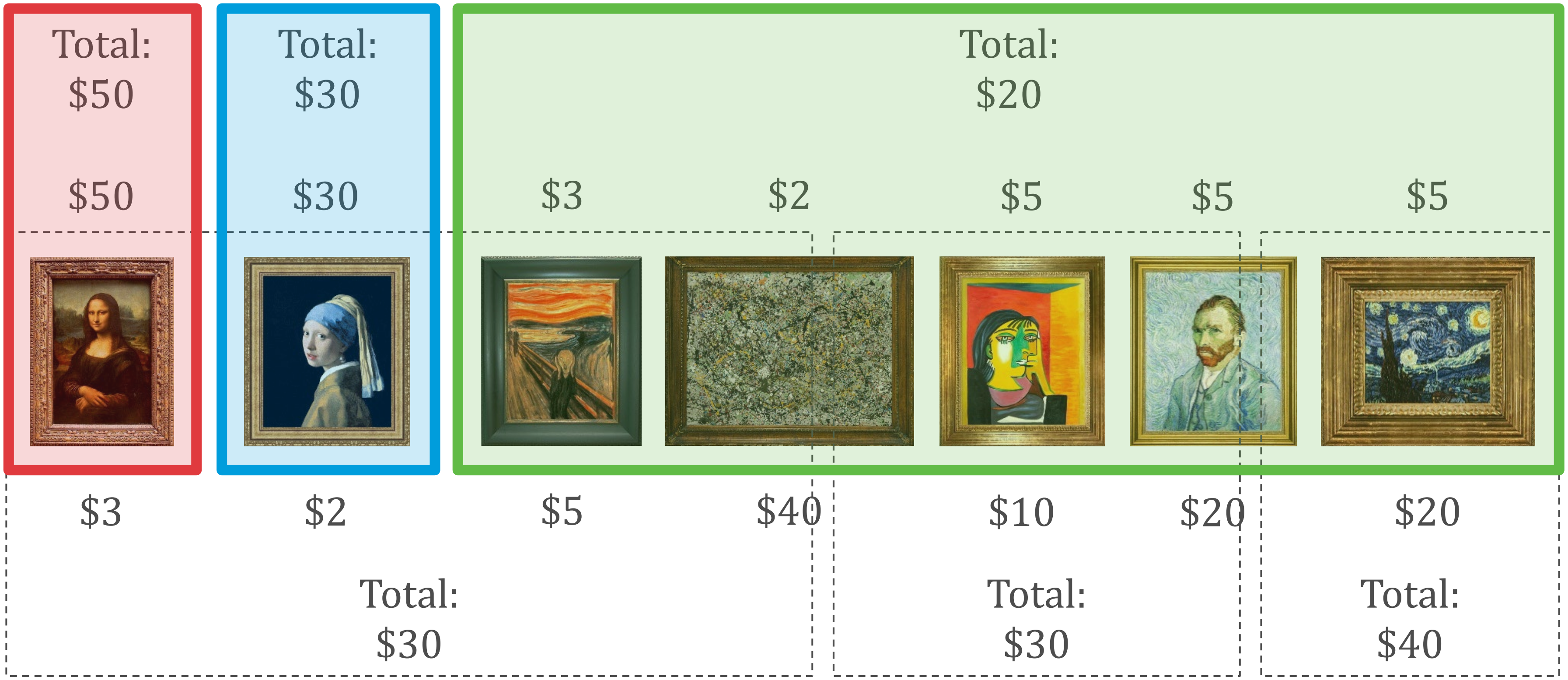
INDIVISIBLE GOODS

- Set G of m goods
- Each good is indivisible
- Players $N = \{1, \dots, n\}$ have valuations V_i for bundles of goods
- Valuations are additive if for all $S \subseteq G$ and $i \in N, V_i(S) = \sum_{g \in S} V_i(g)$
- Assume additivity unless noted otherwise
- An allocation is a partition of the goods, denoted $\boldsymbol{A} = (A_1, \dots, A_n)$
- Envy-freeness and proportionality are infeasible!

MAXIMIN SHARE GUARANTEE



MAXIMIN SHARE GUARANTEE



MAXIMIN SHARE GUARANTEE

- Maximin share (MMS) guarantee of player i:
$$\max_{X_1,\ldots,X_n} \min_j V_i(X_j)$$
- An MMS allocation is such that $V_i(A_i)$ is at least i's MMS guarantee for all $i \in N$
- For $n = 2$ an MMS allocation always exists
- Theorem: $\forall n \geq 3$ there exist additive valuation functions that do not admit an MMS allocation

COUNTEREXAMPLE FOR $n = 3$

17	25	12	1
2	22	3	28
11	0	21	23

3 ways of dividing these numbers into 3 subsets of 4 numbers such that each subset adds up to 55

COUNTEREXAMPLE FOR $n = 3$

1	1	1	1
1	1	1	1
1	1	1	1

$\times 10^6 +$

17	25	12	1
2	22	3	28
11	0	21	23

$\times 10^3 +$

3	-1	-1	-1
0	0	0	0
0	0	0	0

Player 1

3	-1	0	0
-1	0	0	0
-1	0	0	0

Player 2

3	0	-1	0
0	0	-1	0
0	0	0	-1

Player 3

APPROXIMATE ENVY-FREENESS

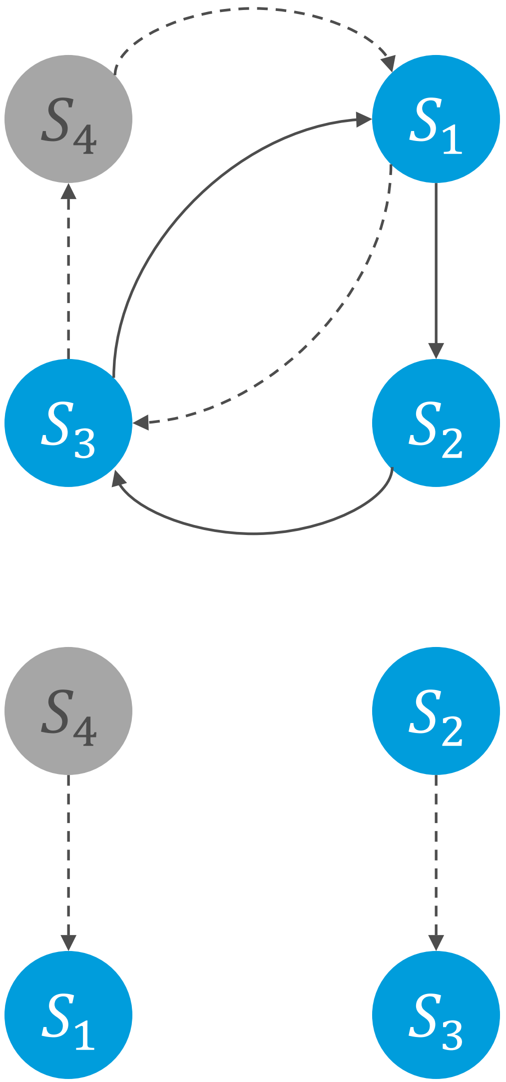
- Assume general monotonic valuations, i.e., for all $S \subseteq T \subseteq G, V_i(S) \leq V_i(T)$
- An allocation $A_1, \dots, A_n$ is envy free up to one good (EF1) if and only if $\forall i, j \in N, \exists g \in A_j$ s.t. $v_i(A_i) \geq v_i\left(A_j \setminus \{g\}\right)$
- Theorem: An EF1 allocation exists and can be found in polynomial time

PROOF OF THEOREM

- A partial allocation is an allocation of a subset of the goods
- Given a partial allocation $\boldsymbol{A}$, we have an edge (i, j) in its envy graph if i envies j
- Lemma: An EF1 partial allocation $\boldsymbol{A}$ can be transformed in polynomial time into an EF1 partial allocation $\boldsymbol{B}$ of the same goods with an acyclic envy graph

PROOF OF LEMMA

- If graph has a cycle C, shift allocations along C to obtain $\boldsymbol{A}'$; clearly EF1 is maintained
- #edges in envy graph of $\boldsymbol{A}'$ decreased:
  - Same edges between $N \setminus C$
  - Edges from $N \setminus C$ to C shifted
  - Edges from C to $N \setminus C$ can only decrease
  - Edges inside C decreased
- Iteratively remove cycles ■

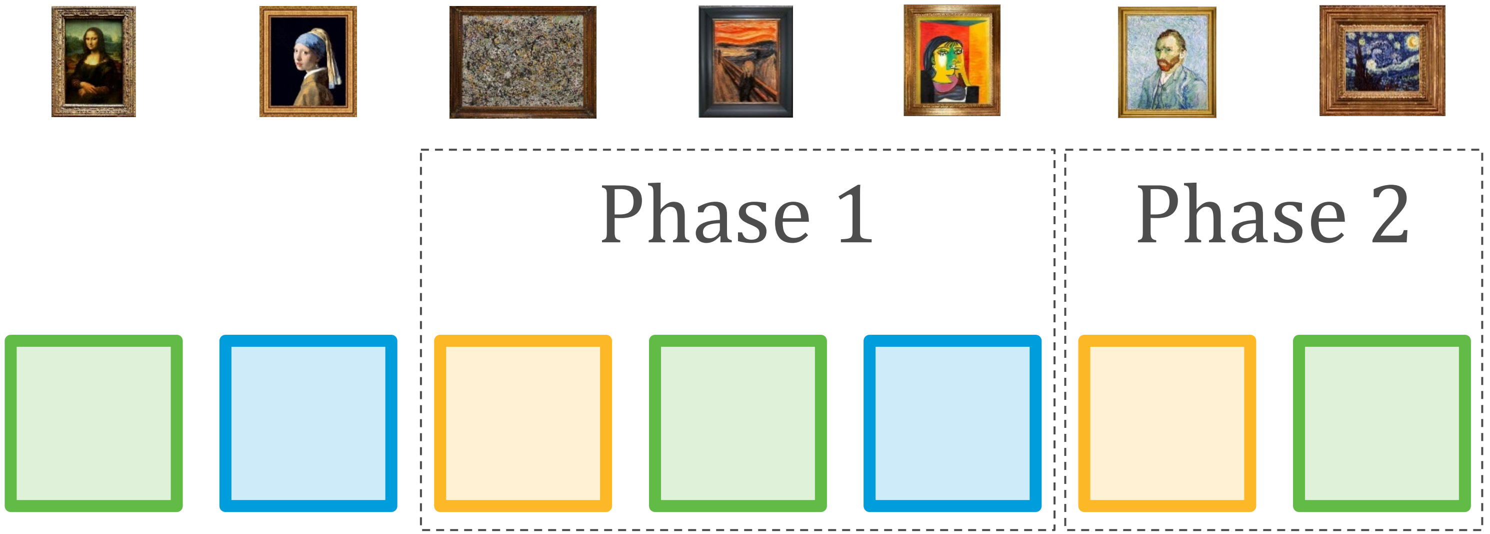


PROOF OF THEOREM

- Maintain EF1 and acyclic envy graph
- In round 1, allocate good g_1 to arbitrary player; envy graph is acyclic and EF1
- $g_1, \dots, g_{k-1}$ are allocated in acyclic and EF1 allocation $\boldsymbol{A}$
- Derive $\boldsymbol{B}$ by allocating g_k to source i
- $V_j(B_j) = V_j(A_j) \geq V_j(A_i) = V_j(B_i \setminus \{g_k\})$
- Use lemma to eliminate cycles ■

ROUND ROBIN

- Let us return to additive valuations
- Now proving the existence of an EF1 allocation is trivial
- A round-robin allocation is EF1:



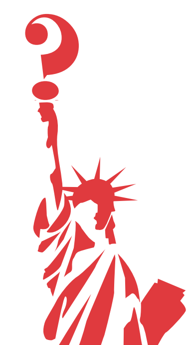
EFFICIENCY AND FAIRNESS

- An allocation $\boldsymbol{A}$ is Pareto efficient if there is no allocation $\boldsymbol{A}'$ such that $V_i(A_i') \geq V_i(A_i)$ for all $i \in N$, and $V_j(A_j') > V_j(A_j)$ for some $j \in N$

Poll

Which of the following rules is Pareto efficient?

- Round Robin
- Max utilitarian social welfare
- Both
- Neither



MAXIMUM NASH WELFARE

- The Nash welfare of an allocation $\boldsymbol{A}$ is the product of values

$$\mathrm{NW}(\boldsymbol{A}) = \prod_{i \in N} V_i(A_i)$$

- The maximum Nash welfare (MNW) solution chooses an allocation that maximizes the Nash welfare
- For ease of exposition we ignore the case of $\mathrm{NW}(\boldsymbol{A}) = 0$ for all $\boldsymbol{A}$
- Theorem: Assuming additive valuations, the MNW solution is EF1 and Pareto efficient

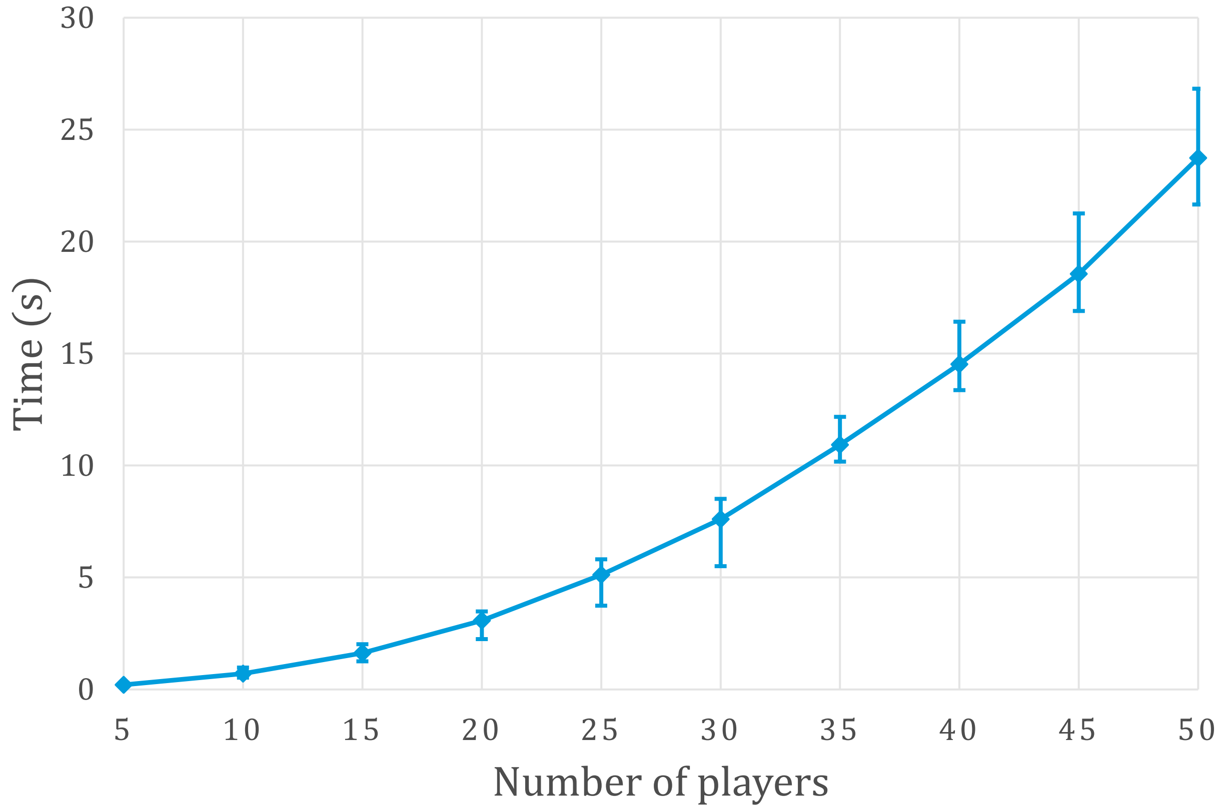
PROOF OF THEOREM

- Efficiency is obvious, so we focus on EF1
- Assume for contradiction that i envies j by more than one good
- Let $g^\star \in \text{argmin}_{g \in A_j} V_j(g)/V_i(g)$
- Move $g^\star$ from j to i to obtain $\boldsymbol{A}'$, we will show that $\text{NW}(\boldsymbol{A}') > \text{NW}(\boldsymbol{A})$
- It holds that $V_k(A_k) = V_k(A'_k)$ for all $k \neq i, j$, $V_i(A'_i) = V_i(A_i) + V_i(g^\star)$, and $V_j(A'_j) = V_j(A_j) - V_j(g^\star)$

PROOF OF THEOREM

- $\frac{\mathrm{NW}(A')}{\mathrm{NW}(A)} > 1 \Leftrightarrow \left[1 - \frac{V_j(g^\star)}{V_j(A_j)}\right]\left[1 + \frac{V_i(g^\star)}{V_i(A_i)}\right] > 1 \Leftrightarrow$ $\frac{V_j(g^\star)}{V_i(g^\star)}[V_i(A_i) + V_i(g^\star)] < V_j(A_j)$
- Due to our choice of $g^\star$,
$$\frac{V_j(g^\star)}{V_i(g^\star)} \leq \frac{\sum_{g\in A_j} V_j(g)}{\sum_{g\in A_j} V_i(g)} = \frac{V_j(A_j)}{V_i(A_j)}$$
- Due to EF1 violation, we have
$$V_i(A_i) + V_i(g^\star) < V_i(A_j)$$
- Multiply the last two inequalities to get the first ■

TRACTABILITY OF MNW



[Caragiannis et al., 2016]

INTERFACE


THE BASICS
ALICE'S EVALUATIONS ✓
Alice, use the sliders to assign values to each of the items below. All of your values must sum to 1000. You can use the rescale button to automatically adjust your values to add up to 1000.
Gold Ring
59
Diamond Ring
145
Pearl Necklace
116
Ruby Earrings
265
Gold Watch
80
Silver Bracelet
335
RESET
RESCALE
CONTINUE
Current Total: 1000
Target: 1000
BOB'S EVALUATIONS
CLAIRE'S EVALUATIONS
RESULTS

AN OPEN PROBLEM

- An allocation $A_1, \ldots, A_n$ is envy free up to any good (EFX) if and only if
  $\forall i, j \in N, \forall g \in A_j,\ v_i(A_i) \geq v_i(A_j \setminus \{g\})$
- Strictly stronger than EF1, strictly weaker than EF
- An EFX allocation exists for two players with monotonic valuations (easy) and for three players with additive valuations (very hard)
- Existence is an open problem for $n \geq 4$ players with additive valuations

BIBLIOGRAPHY

E. Budish. The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes. Journal of Political Economy, 2011.

R. J. Lipton, E. Markakis, E. Mossel and A. Saberi. On Approximately Fair Allocations of Indivisible Goods. EC 2004.

D. Kurokawa, A. D. Procaccia and J. Wang. Fair Enough: Guaranteeing Approximate Maximin Shares. Journal of the ACM, 2018.

I. Caragiannis, D. Kurokawa, H. Moulin, A. D. Procaccia, N. Shah and J. Wang. The Unreasonable Fairness of Maximum Nash Welfare. ACM Transactions on Economics and Computation, 2019.