Economics and Computation (Spring 2025)
Assignment #2
— Solutions —

Due: 2/26/2025 11:59pm ET

Problem 1: The price of anarchy

Consider the following scheduling game. The players $N = \{1, \ldots, n\}$ are associated with tasks, each with weight w_i. There is also a set M of m machines. Each player chooses a machine to place their task on, that is, the strategy space of each player is M. A strategy profile induces an assignment $A : N \to M$ of players (or tasks) to machines; the *cost* of player i is the total load on the machine to which i is assigned: $\ell_{A(i)} = \sum_{j \in N:\ A(j)=A(i)} w_j$. Our objective function is the *makespan*, which is the maximum load on any machine: $\text{cost}(A) = \max_{\mu \in M} \ell_\mu$. It is known that scheduling games always have pure Nash equilibria.

1. **[15 points]** Let G be a scheduling game with n tasks of weight $w_1, \ldots, w_n$, and m machines. Let $A : N \to M$ be a Nash equilibrium assignment. Prove that

$$\text{cost}(A) \leq \left(2 - \frac{2}{m+1}\right) \cdot \text{opt}(G).$$

That is, the price of anarchy is at most $2 - 2/(m+1)$.

Solution: The solution is copied from the proof of Theorem 20.5 in the AGT book [2].

Let j^* be the machine with highest load under the given Nash equilibrium assignment A, and let i^* be a task of smallest weight assigned to this machine. Without loss of generality there are two tasks assigned to j^* as otherwise $\text{cost}(A) = \text{OPT}(G)$, and the bound trivially follows. Thus $w_{i^*} \leq \frac{1}{2}\text{cost}(A)$.

Suppose there is a machine $j \neq j^*$ with load less than $\ell_{j^*} - w_{i^*}$. Then moving i^* from j^* to j would decrease the cost of this task. Hence, as A is a Nash equilibrium, it holds that

$$\ell_j \geq \ell_{j^*} - w_{i^*} \geq \text{cost}(A) - \frac{1}{2}\text{cost}(A) = \frac{1}{2}\text{cost}(A).$$

Now observe that the cost of an optimal assignment cannot be smaller than the average load across all machines, so

$$\text{OPT}(G) \geq \frac{\sum_i w_i}{m} = \frac{\sum_j \ell_j}{m} \geq \frac{\text{cost}(A) + \frac{1}{2}\text{cost}(A)(m-1)}{m} = \frac{(m+1)\text{cost}(A)}{2m}.$$

It follows that

$$\text{cost}(A) \le \frac{2m}{m+1} \cdot \text{OPT}(G) = \left(2 - \frac{2}{m+1}\right) \cdot \text{OPT}(G).$$

2. **[10 points]** Prove that the upper bound of part (a) is tight, by constructing an appropriate family of scheduling games for each $m \in \mathbb{N}$.

   Solution: Consider a game G with $n = 2m$: m small jobs with weight 1, and m large jobs with weight m. Clearly $\text{OPT}(G) = m+1$, by putting one small job and one large job on each machine.

   Now, consider the assignment A that puts two large jobs on machine 1, all the small jobs on machine 2, and one large job on each of the machines $3, \ldots, m$. It holds that $\text{cost}(A) = 2m$ and A is a Nash equilibrium.

Problem 2: Voting rules

[10 points] When the number of alternatives is m, a *positional scoring rule* is defined by a score vector $(s_1, \ldots, s_m)$ such that $s_k \ge s_{k+1}$ for all $k = 1, \ldots, m-1$. Each voter gives s_k points to the alternative they rank in position k, and the points are summed over all voters. We discussed two examples of positional scoring rules: plurality, defined by the vector $(1, 0, \ldots, 0)$, and Borda, defined by the vector $(m-1, m-2, \ldots, 0)$. Another common example is *veto*, defined by the vector $(1, \ldots, 1, 0)$.

For the case of $m = 3$, prove that any positional scoring vector with $s_2 > s_3$ is *not* Condorcet consistent.

Hint: It is possible to do this via a single preference profile that includes 7 voters.

Solution: Consider the following preference profile for 7 voters among three alternatives a_1, a_2, a_3:

Voter Group	**Ranking**
3 voters	$a_1 \succ a_2 \succ a_3$
2 voters	$a_2 \succ a_3 \succ a_1$
1 voter	$a_2 \succ a_1 \succ a_3$
1 voter	$a_3 \succ a_1 \succ a_2$

We want to show that no scoring vector with $s_2 > s_3$ elects the Condorcet winner, which is a_1, since a_1 beats a_2 and a_3 in direct comparisons with 4 out of 7 votes in both cases.

Let (s_1, s_2, s_3) be an arbitrary scoring vector. The points each alternative receives are:

Candidate	**Points**
a_1	$3s_1 + 2s_2 + 2s_3$
a_2	$3s_1 + 3s_2 + 1s_3$
a_3	$1s_1 + 2s_2 + 4s_3$

We see that

$$\text{score}(a_1) - \text{score}(a_2) = s_3 - s_2 < 0,$$

since $s_2 > s_3$. Therefore, a_2 is elected winner, but a_1 is the Condorcet winner, so the positional scoring rule defined by (s_1, s_2, s_3) is not Condorcet consistent.

Problem 3: The epistemic approach to voting

[10 points] Suppose that there is a true ranking of m alternatives, each of n voters evaluates all pairs of alternatives according to the Condorcet noise model (Lecture 6, slide 5) with $p > 1/2$, and these comparisons are aggregated into a voting matrix. Prove that the output of the Kemeny rule applied to this voting matrix coincides with the true ranking with probability that goes to 1 as n goes to infinity.

Hint: Use the Condorcet Jury Theorem (or the law of large numbers).

Solution: According to the Condorcet noise model's assumptions, let the true ranking of the m alternatives be $a_1 \succ a_2 \succ \cdots \succ a_m$. We know for any $i < j$, each voter independently votes for a_i over a_j w.p. $p > 0.5$ (in other words, they vote 'correctly' more than half the time, and the 'wrong' votes are the noise introduced).

Where $i < j$, let V_{ij} be the entry in the voting matrix corresponding to the number of voters who voted for a_i over a_j. By the Condorcet Jury Theorem, the probability that $V_{ij} > 1/2$ goes to 1 as n goes to infinity. By taking a union bound over all $i < j$, we have that

$$\lim_{n\to\infty} \Pr[\forall i < j,\ V_{ij} > 1/2] = 1.$$

Hence, the ranking that minimizes the sum of Kendall tau distances will be the true ranking w.p. approaching 1.

Problem 4: Strategic manipulation in elections

We saw in class a proof sketch of the Gibbard-Satterthwaite Theorem for the special case of strategyproof and neutral voting rules with $m \geq 3$ and $m \geq n$. That proof relied on two key lemmas. In this problem, you will prove the two lemmas and formalize the theorem's proof for this special case.

Prove the following statements.

1. **[10 points]** Let f be a strategyproof voting rule, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be a preference profile, and $f(\boldsymbol{\sigma}) = a$. If $\boldsymbol{\sigma}'$ is a profile such that $[a \succ_{\sigma_i} x \ \Rightarrow\ a \succ_{\sigma'_i} x]$ for all $x \in A$ and $i \in N$, then $f(\boldsymbol{\sigma}') = a$.

   Solution: The proofs of all three parts are copied from Svensson [1]. Suppose first that $\sigma_i = \sigma'_i$ for $i > 1$. Let $f(\succ_{\sigma'_1}, \succ_{\sigma'_{-1}}) = b$. From strategyproofness it follows that $a \succeq_{\sigma_1} b$, and hence from the assumption of the lemma, $a \succeq_{\sigma'_1} b$. Strategyproofness also implies that $b \succeq_{\sigma'_1} a$, and because preferences are strict it follows that $a = b$. The lemma now follows after repeating this argument while changing the preferences for only $i = 2$, then $i = 3$, etc.

2. **[10 points]** Let f be a strategyproof and onto voting rule. Furthermore, let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be a preference profile and $a, b \in A$ such that $a \succ_{\sigma_i} b$ for all $i \in N$. Then $f(\boldsymbol{\sigma}) \neq b$.

   Hint: use part (a).

   Solution: Suppose that $f(\boldsymbol{\sigma}) = b$. Since f is onto there is a profile $\boldsymbol{\sigma}'$ such that $f(\boldsymbol{\sigma}') = a$. Let $\boldsymbol{\sigma}''$ be such that for all $i \in N$, $a \succ_{\sigma''_i} b \succ_{\sigma''_i} x$ for all $x \in A \setminus \{a, b\}$, and the rest of the alternatives are ranked identically to σ_i. By strong monotonicity (part (a)), $b = f(\boldsymbol{\sigma}) = f(\boldsymbol{\sigma}'')$ and $a = f(\boldsymbol{\sigma}') = f(\boldsymbol{\sigma}'')$, which is a contradiction. Hence $f(\boldsymbol{\sigma}) \neq b$.

3. **[10 points]** Let m be the number of alternatives and n be the number of voters, and assume that $m \geq 3$ and $m \geq n$. Furthermore, let f be a strategyproof and neutral voting rule. Then f is dictatorial.

   Important note: There are many proofs of the full version of the Gibbard-Satterthwaite Theorem; *here the task is specifically to formalize the proof sketch we did in class.*

   Solution: For this part of the proof it is convenient to define the preferences of each $i \in N$ via a utility function u_i such that for $x, y \in A$, $x \succ_{\sigma_i} y$ if and only if $u_i(x) > u_i(y)$. Therefore, $f(\boldsymbol{u})$ is well defined. We will also denote $A = \{a_1, \ldots, a_m\}$.

   For each $i \in N$, let

$$u_i(a_j) = \begin{cases} n + i - j & i \leq j \leq n \\ i - j & j < i \\ n - j & j > n \end{cases}$$

   That is, the ranking of $a_1, \ldots, a_n$ is shifted, and all other alternatives are ranked below them. By Pareto optimality (part b), $f(\boldsymbol{u}) = a_j$ for some $j \leq n$. Assume w.l.o.g. that $f(\boldsymbol{u}) = a_1$. Let $\boldsymbol{u}'$ be defined as follows:

$$\begin{aligned} &u'_1(a_1) = n + 2 \text{ and } u'_1(a_n) = n + 1, \\ &u'_i(a_n) = n + 2 \text{ and } u'_i(a_1) = n + 1 \text{ for } i > 1, \\ &u'_i(a_j) = u_i(a_j) \text{ otherwise} \end{aligned}$$

   Hence all voters consider the alternatives a_1 and a_n to be better than the other alternatives. Also note that the ranking of a_1 and a_n is the same in the profiles $\boldsymbol{u}$ and $\boldsymbol{u}'$; and in $\boldsymbol{u}'$, a_1 and a_n are both ranked above other alternatives. Hence by strong monotonicity (part (a)), $f(\boldsymbol{u}') = f(\boldsymbol{u}) = a_1$.

   Finally, define profiles $\boldsymbol{u}^k$ for $k = 1, \ldots, n$, where $\boldsymbol{u}^1 = \boldsymbol{u}'$, and

$$u_i^{k+1}(x) = \begin{cases} u_i^k(x) & i \neq k + 1 \\ u_{k+1}^k(x) & i = k + 1 \text{ and } x \in A \setminus \{a_1\} \\ -m & i = k + 1 \text{ and } x = a_1 \end{cases}$$

   By Pareto optimality (part (b)), $f(\boldsymbol{u}^k) \in \{a_1, a_n\}$. But strategyproofness implies that $f(\boldsymbol{u}^k) = a_1$, and hence $f(\boldsymbol{u}^n) = a_1$. In $\boldsymbol{u}^n$, a_1 is ranked at the top by voter 1, and at the bottom by every other voter. Monotonicity (part (a)) implies that a_1 is the winner whenever voter 1 puts a_1 at the top. Neutrality then implies that voter 1 is a dictator.

References

[1] L.-G. Svensson. The proof of the Gibbard-Satterthwaite theorem revisited. Working Paper No. 1999:1, Department of Economics, Lund University, 1999. Available from: http://www.nek.lu.se/NEKlgs/vote09.pdf.

[2] B. Vöcking. Selfish load balancing. In N. Nisan, T. Roughgarden, É. Tardos, and V. Vazirani, editors, *Algorithmic Game Theory*, chapter 20. Cambridge University Press, 2007.