# Economics and Computation (Spring 2025)
# Assignment #1
# — Solutions —

Due: 2/12/2025 11:59pm ET

## Problem 1: Nash equilibrium

Recall the ice cream wars setting from Lecture 1. Suppose that there are $n$ ice cream sellers on a beach which can be modeled as the interval $[0, 1]$ and that each ice cream seller chooses a real number within that interval where they will set up their cart to sell ice cream. The ice cream being sold from all carts is identical, that is, customers are indifferent about which cart's ice cream they buy and will always buy from the seller closest to them. Customers are uniformly distributed along the beach (assume distance only matters to customers in the horizontal direction along the beach's length, the width of the beach is negligible). We saw that when $n = 2$, there is a Nash equilibrium in pure strategies where both sellers are located at $1/2$.

1. **[10 points]** Prove that when $n = 3$, a Nash equilibrium in pure strategies does not exist.

   **Solution:** For $k = 3$, there is no pure strategy Nash equilibrium. Denote the seller positions by $x_1, x_2, x_3$, and assume w.l.o.g. that $x_1 \leq x_2 \leq x_3$.

   If $x_1 = x_2 = x_3$ then all three sellers are getting utility $1/3$. If w.l.o.g. their common position is at most $1/2$ (i.e., $x_1 = x_2 = x_3 \leq 1/2$), then any of the sellers can deviate to 0.51 and get utility greater than 0.49.

   Otherwise, $x_1 < x_3$. If $x_1 < x_2$, then $x_1$ can move closer to $x_2$ and obtain higher utility. Otherwise it holds that $x_1 = x_2 < x_3$, and $x_3$ will gain by moving closer to $x_2$.

2. **[15 points]** Prove that when $n$ is even, a Nash equilibrium in pure strategies exists.

   **Solution:** When $n$ is even, a pure strategy Nash equilibrium is achieved when exactly two players choose each of the positions $\frac{1}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}$. Note that for the two players at position $\frac{k}{n}$, where $k$ is an odd integer such that $1 \leq k \leq n-1$, these two players capture all customers in the interval $\left(\frac{k-1}{n}, \frac{k+1}{n}\right)$, which has a length of $\frac{2}{n}$. Since two players share this interval, each receives a payoff of $\frac{1}{n}$.

   Consider a deviation to another position already occupied by two players. In this case, three players would share the interval of length $\frac{2}{n}$, resulting in a payoff of $\frac{2}{3n}$ for the player, which is less than their original $\frac{1}{n}$.

Now, consider a deviation to an unoccupied position outside the range of existing positions, such as $x < \frac{1}{n}$. The deviating player would serve customers from 0 to $x$ and up to the midpoint between $x$ and $\frac{1}{n}$. However, the length of this interval is less than $\frac{1}{n}$. A similar argument holds for deviations to positions $x > \frac{n-1}{n}$.

Finally, consider a deviation to an unoccupied position between two occupied positions, such as $y$ between $\frac{k}{n}$ and $\frac{k+2}{n}$ for some odd integer $k$ such that $1 \leq k \leq n-3$. The deviating player would serve customers from the midpoint between $\frac{k}{n}$ and $y$ to the midpoint between $y$ and $\frac{k+2}{n}$. The total length of this interval is $\frac{1}{n}$, which is equal to the deviator's payoff before the deviation.

In all cases, a unilateral deviation results in a payoff less than or equal to $\frac{1}{n}$. Therefore, no player has an incentive to deviate, confirming that the proposed configuration is a pure strategy Nash equilibrium.

## Problem 2: Equilibrium computation

In a normal-form game, we say that strategy $s_i$ is *strictly dominated* for player $i$ if there exists a mixed strategy $x_i$ of player $i$ such that $u_i(x_i, \mathbf{s}_{-i}) > u_i(\mathbf{s})$ for all pure strategy profiles $\mathbf{s}_{-i}$ of the other players.

For example, in the following game, M is dominated by $(0.5, 0, 0.5)$:

| | | |
|---|---|---|
| U | (2,0) | (-1,0) |
| M | (0,0) | (0,0) |
| D | (-1,0) | (2,0) |

A process of *iterated elimination of strictly dominated strategies* starts by identifying a strictly dominated strategy, if one exists. That strategy is then removed from the strategy set of the relevant player, and the process is repeated for the smaller game. The process terminates when strictly dominated strategies no longer exist.

1. **[25 points]** Prove that if $\mathbf{x}$ is a mixed strategy Nash equilibrium, and $x_i(s_i) > 0$, then $s_i$ cannot be eliminated in a process of iterated elimination of strictly dominated strategies.

   **Guidance**: Use induction on the order of elimination. If $s_i$ was eliminated, it is strictly dominated by a mixed strategy $x'_i$; in $x_i$, transfer the probability weight of $s_i$ to $x'_i$, thus obtaining a strictly better mixed strategy that violates the assumption that $\mathbf{x}$ is a Nash equilibrium.

   **Solution:** Following the guidance, we proceed by induction on the order of elimination.

   For the base of the induction, initially all strategies are available to all players. Suppose for contradiction that $x_i(s_i^\star) > 0$ but $s_i^\star$ is eliminated in the first round of the iterated elimination process. Let $x'_i$ be the (possibly) mixed strategy that strictly dominates $s_i^\star$. We construct a new mixed strategy $x''_i$, defined for any pure strategy $s_i \neq s_i^\star$ by

$$x''_i(s_i) = x_i(s_i) + x_i(s_i^\star) \cdot x'_i(s_i),$$

   and for $s_i^\star$ by

$$x''_i(s_i^\star) = x_i(s_i^\star) \cdot x'_i(s_i^\star).$$

Note that $x_i''$ can be seen as transferring the probability weight of $x_i(s_i^\star)$ to $x_i'$. Therefore, for all pure strategy profiles $\mathbf{s}_{-i}$,

$$u_i(x_i'', \mathbf{s}_{-i}) - u_i(x_i, \mathbf{s}_{-i}) = x_i(s_i^\star) \cdot (u_i(x_i', \mathbf{s}_{-i}) - u_i(s_i^\star, \mathbf{s}_{-i})) > 0,$$

where the inequality holds because $x_i(s_i^\star) > 0$ and $s_i^\star$ is strictly dominated by $x_i'$. Now, since the inequality holds pointwise for each $\mathbf{s}_i$, we conclude that

$$u_i(x_i'', \mathbf{x}_{-i}) - u_i(x_i, \mathbf{x}_{-i}) > 0,$$

in contradiction to the assumption that $\mathbf{x}$ is a Nash equilibrium.

For the induction step, assume that $k$ strategies have already been eliminated, and we want to show that elimination step $k+1$ does not eliminate a strategy in the support of $\mathbf{x}$. By the induction assumption, the first $k$ elimination steps have not eliminated any strategies in the support of $\mathbf{x}$. Therefore, $\mathbf{x}$ is still a Nash equilibrium in the reduced game, and we can repeat the exact same analysis as in the base case.

2. **[10 points]** For the case of two players, design an algorithm that performs iterated elimination of strictly dominated strategies in time that is polynomial in the size of the strategy set.

   **Hint**: Use linear programming. You may assume that LPs can be solved in polynomial time and, although this is a bit messy, you may include strict inequalities in your LP.

   **Solution:** The following feasibility LP determines whether $s_1^\star$ is strictly dominated by identifying a dominating mixed strategy $x_1$; the variables are $x_1(s_1)$ for each pure strategy $s_1 \in S$. (Assume for ease of notation that both players share the same strategy set $S$.)

$$\sum_{s_1 \in S} x_1(s_1) u_1(s_1, s_2) > u_1(s_1^\star, s_2) \quad \forall s_2 \in S$$

$$x_1(s_1) \geq 0 \quad \forall s_1 \in S$$

$$\sum_{s_1 \in S} x_1(s_1) = 1$$

   If such an $\mathbf{x}_1$ is feasible, then $s_1^\star$ is strictly dominated, and an analogous LP works for player 2.

   For running time, there are at most $2|S|$ elimination steps, and each elimination step involves checking at most $2|S|$ strategies using an LP that can be solved in polynomial time.

Taken together, these two facts suggest that iterated elimination of dominated strategies can be a useful *preprocessing* step when attempting to compute a mixed Nash equilibrium.

## Problem 3: Extensive-Form Games

1. **[10 points]** Consider a 2-player game in normal form. Let $(x_1, x_2)$ be a (possibly mixed) Nash equilibrium. Show that if $(x_1^\star, s_2)$ is a strong Stackelberg equilibrium (where player 1 is the leader) then $u_1(x_1^\star, s_2) \geq u_1(x_1, x_2)$.

**Solution:** If $(x_1^\star, s_2)$ is a strong Stackelberg equilibrium, then $x_1^\star$ maximizes the quantity $\max_{s_2 \in B_2(x)} u_1(x, s_2)$, considering all mixed strategies $x$. By the definition of a strong Stackelberg equilibrium:

$$u_1(x_1^\star, s_2) = \max_{s_2 \in B_2(x_1^\star)} u_1(x_1^\star, s_2) \geq \max_{s_2 \in B_2(x_1)} u_1(x_1, s_2)$$

so it suffices to show that $u_1(x_1, x_2) \leq \max_{s_2 \in B_2(x_1)} u_1(x_1, s_2)$.

Since $(x_1, x_2)$ is a NE, we claim that $x_2$ must have support entirely in $B_2(x_1)$. Assume for contradiction that $x_2$ contains in its support a strategy that itself is not a best response to $x_1$. Then, player 2 would be better off decreasing the attributed probability of that strategy and increasing the attributed probability to a best response to $x_1$, which is a contradiction because $(x_1, x_2)$ is a NE.

Thus, we get:

$$u_1(x_1, x_2) = \sum_{s_2 \in B_2(x_1)} x_2(s_2) u_1(x_1, s_2) \leq \max_{s_2 \in B_2(x_1)} u_1(x_1, s_2),$$

as desired.

2. **[10 points]** Prove that in the game of chess, precisely one of the following holds: white has a strategy that guarantees a win no matter how black plays, black has a strategy that guarantees a win no matter how white plays, or each player has a strategy that guarantees a tie no matter how the other plays.

   **Hint**: Backward induction.

   **Solution:** Solve for the subgame-perfect equilibrium for the game of chess using backward induction. Since the payoffs at the leaves are -1 (black wins), 0 (draw) or 1 (white wins), these payoffs propagate upwards through the backward induction process until the root is labeled with precisely one of them.

   Assume that the value at the root is 1, meaning that there is a subgame-perfect equilibrium where white wins. It follows that white has a winning strategy: given white's strategy, no matter what strategy black plays, they cannot achieve a better outcome. Similar reasoning applies to the cases where the value at the root is 0 (each player can guarantee a tie) and -1 (black has a winning strategy).

## Problem 4: Formulate a Research Question

**[20 points]** Formulate a research question that is relevant to one of the topics covered in this assignment: game theory, equilibrium computation, and extensive-form games. Refer to this document for guidelines.

During the process of formulating your question, keep track of your findings in a "research journal." At a minimum, it should include brainstorming ideas for questions and notes on relevant papers that you have identified.

Please submit the following deliverables:

1. Your research question.
2. A brief explanation of why it satisfies each of the following criteria:
   (a) Relevant: Which course topics is the question related to?
   (b) Nontrivial: What is an immediate way of attempting to answer the question and why does it fail?

   **Note:** This item is optional in Assignment #1.
   (c) Feasible: How would you tackle the question if you had the entire semester?

   **Note:** This item is optional in Assignment #1.
   (d) Novel: List the 1–3 most closely related papers that you have identified in your literature review and explain how your question differs.
3. Append your research journal to the PDF that contains your solutions. The research journal will not be graded; it is there to show your work.