ECONOMICS & COMPUTATION

Spring 2025 | Lecture 9

Strategyproof Approximation Algorithms

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WHEN VCG FALLS SHORT

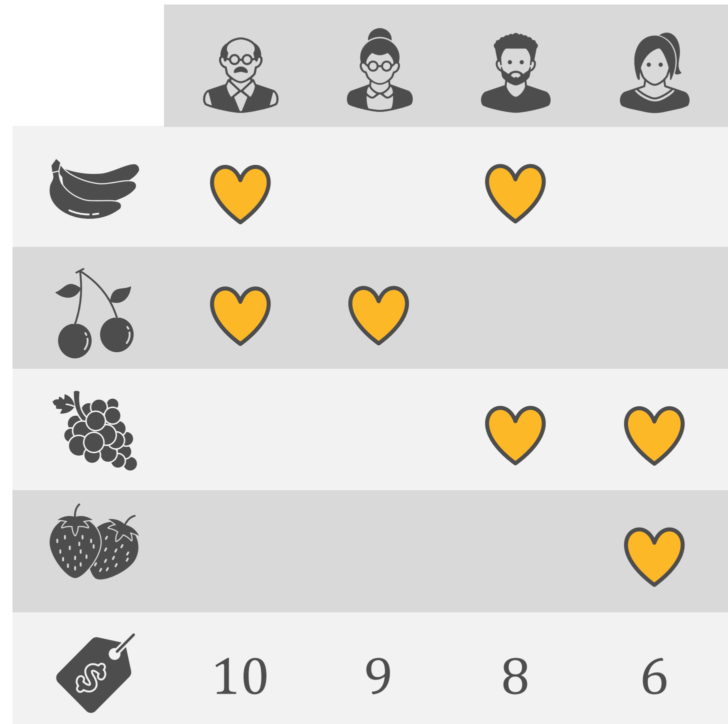
- VCG is an amazing mechanism
- Its Achilles heel, though, is in computing

$$f(\boldsymbol{v}) \in \text{argmax}_{x \in A} \sum_{i \in N} v_i(x)$$

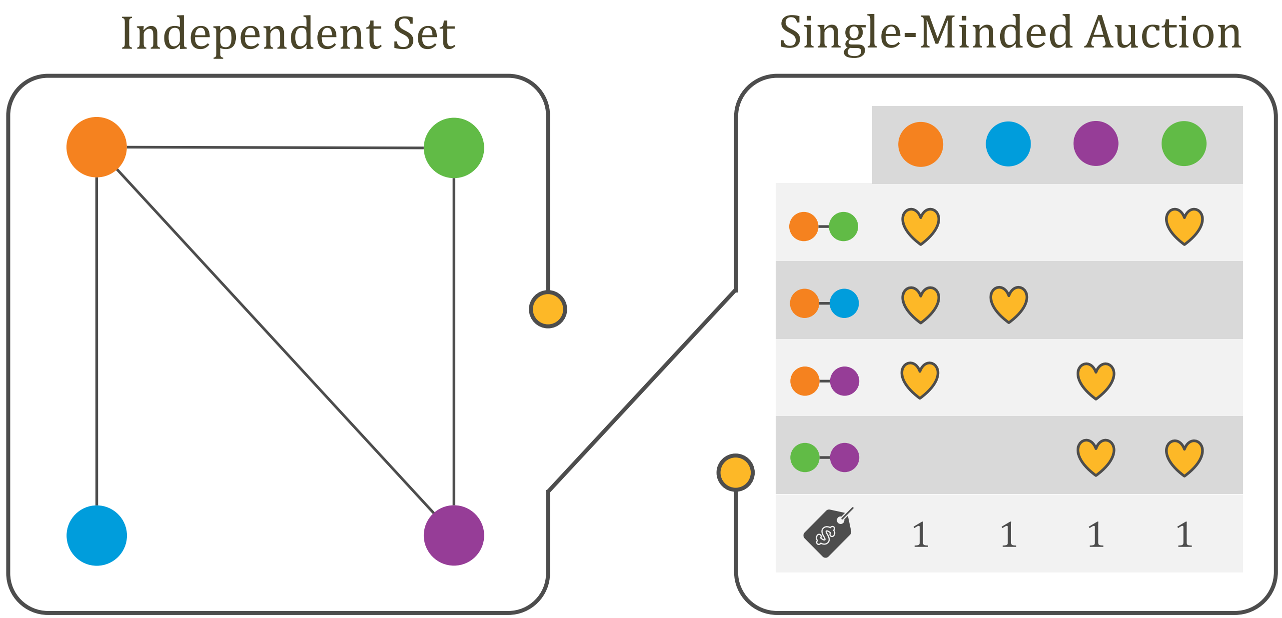
- What do we do if this optimization problem is computationally hard?
- We could solve it approximately, but then we would lose strategyproofness!
- Our goal: approximation and strategyproofness

SINGLE-MINDED AUCTIONS

A set G, $|G| = m$, of goods to allocate. Every player $i \in N$ has a target bundle $T_i \subseteq G$, and has value $v_i(S) = w_i \geq 0$ for $T_i \subseteq S$ and $v_i(S) = 0$ otherwise.



COMPUTATIONAL HARDNESS



- Theorem: Maximizing welfare in single-minded auctions is NP-hard
- Proof:
  - Immediate reduction from Independent Set
  - The set of items is E, there's a player for each vertex, desired bundle is adjacent edges and $w_i = 1$ for all i ■

GREEDY MECHANISM

The greedy single-minded auction for selling a set of items G receives bids (T_i, w_i) for all $i \in N$, and is defined by

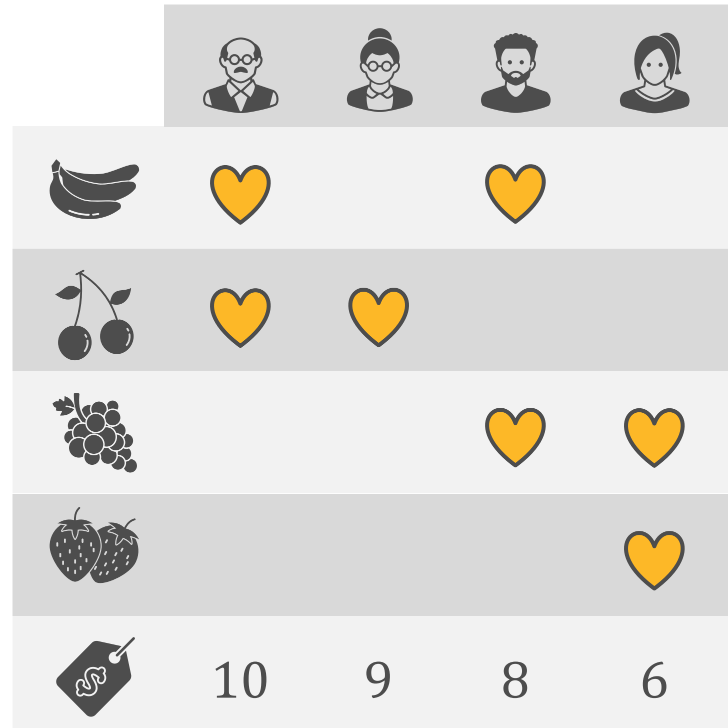
- Allocation rule: sort bids in order of decreasing w_i, breaking ties arbitrarily, and accept bids greedily when they are still feasible
- Payment rule: each allocated player pays the critical value, i.e., the smallest w_i' such that the bid (T_i, w_i') would be accepted

GREEDY MECHANISM: EXAMPLE

Poll 1

What is the payment of the rightmost player?

- 0
- 2
- 3
- 6

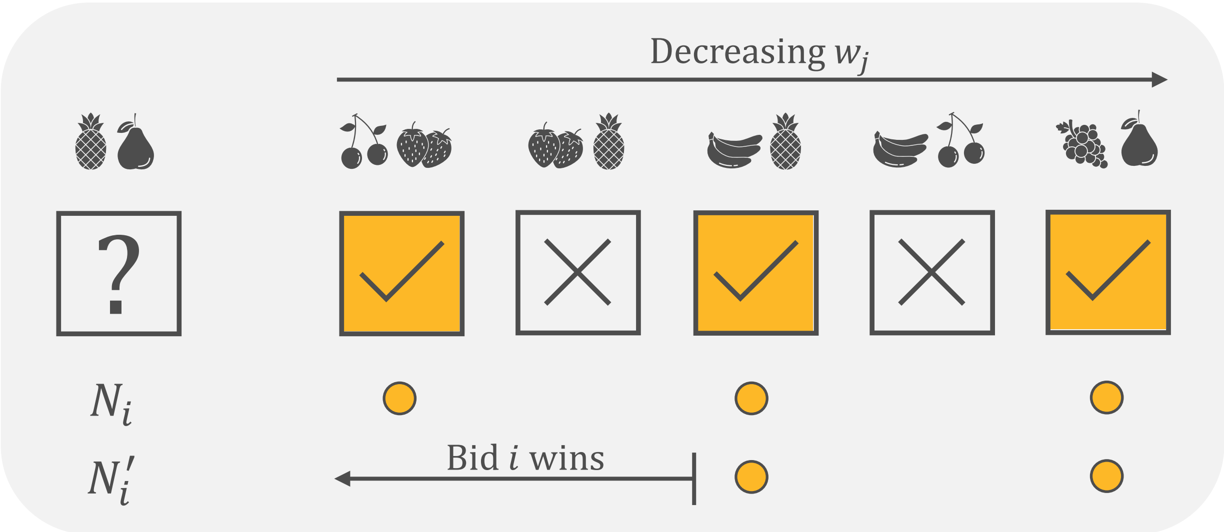


CRITICAL VALUES

- Let N_i be the set of winners if i is removed
- Define the conflict set of i as

$$N_i'(T_i) = \{j \in N_i : T_i \cap T_j \neq \emptyset\}$$

- Lemma: Fixing the bids from others, the critical value of i is $w_i^c = \max_{j \in N_i'(T_i)} w_j$



STRATEGYPROOFNESS

- **Theorem:** The greedy single-minded auction is strategyproof
- **Proof:**
  - It isn't useful to report T_i' that doesn't contain T_i, so assume $T_i \subseteq T_i'$
  - For any such T_i', the allocation is monotone weakly increasing in the reported value w_i'
  - From the lemma, i is allocated T_i' at price $w_i^c(T_i')$ if and only if $w_i' \geq w_i^c(T_i')$, so $w_i' = w_i$ is optimal
  - (T_i, w_i) is weakly preferred to (T_i', w_i) for any $T_i \subseteq T_i'$ because $w_i^c(T_i) \leq w_i^c(T_i')$ ■

APPROXIMATION

- An algorithm for a maximization problem is a c-approximation algorithm for $c \leq 1$ if for every instance $\mathcal{I}$, $ALG(\mathcal{I}) \geq c \cdot OPT(\mathcal{I})$
- An algorithm for a minimization problem is a c-approximation algorithm for $c \geq 1$ if for every instance $\mathcal{I}$, $ALG(\mathcal{I}) \leq c \cdot OPT(\mathcal{I})$

Poll 2

What is the approximation ratio of the greedy single-minded auction?

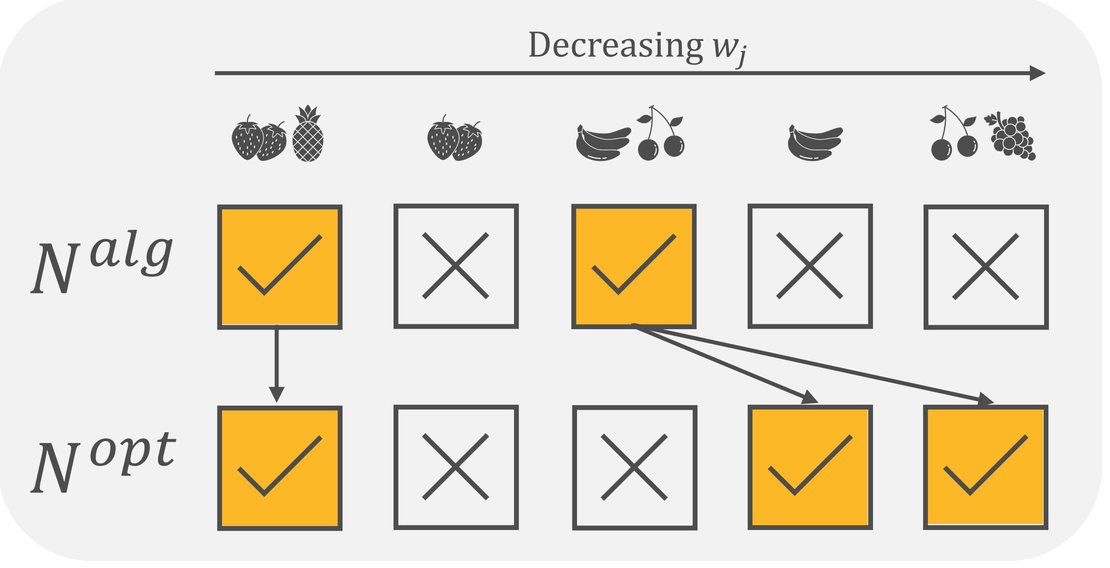
○ $1/2$ ○ $1/3$ ○ $\Theta(1/\log n)$ ○ $\Theta(1/n)$

APPROXIMATION

- Theorem: The greedy single-minded auction is a $1/d$-approximation algorithm, where d is the maximum size of any target bundle
- A variant of the greedy auction where players are ordered by $w_i/\sqrt{|T_i|}$ gives a $1/\sqrt{m}$-approximation for m items
- A better approximation is NP-hard

PROOF OF THEOREM

- Let N^{alg} denote the players allocated under the algorithm, and N^{opt} those allocated under OPT
- For $i \in N^{alg}$, if $i \notin N^{opt}$, let N_i be the set of players $j \in N^{opt}$ such that $w_j \leq w_i$ and $T_i \cap T_j \neq \emptyset$, and if $i \in N^{opt}$, let $N_i = \{i\}$



PROOF OF THEOREM

- It holds that

$$\sum_{j \in N_i} w_j \leq \sum_{j \in N_i} w_i \leq d \cdot w_i$$

- In addition,

$$N^{opt} = \bigcup_{i \in N^{alg}} N_i$$

- We conclude that

$$OPT = \sum_{j \in N^{opt}} w_j \leq \sum_{i \in N^{alg}} \sum_{j \in N_i} w_j \leq d \sum_{i \in N^{alg}} w_i$$

$$= d \cdot ALG \blacksquare$$