ECONOMICS & COMPUTATION

Spring 2025 | Lecture 8

The VCG Mechanism

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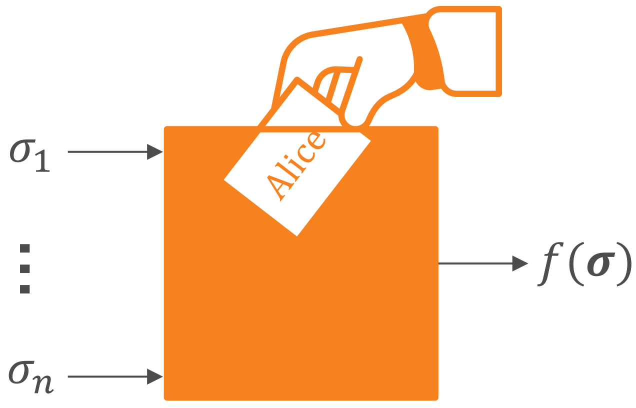
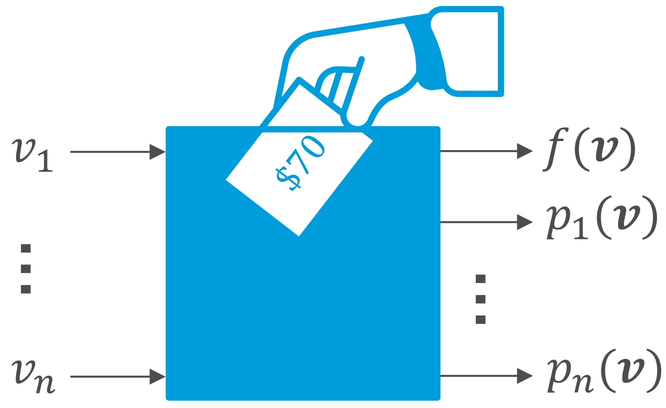
MECHANISMS WITH PAYMENTS

- Consider a set of players $N = \{1, \dots, n\}$ and a set of alternatives A
- Each player $i \in N$ has a valuation function $v_i: A \to \mathbb{R}$
- Players have quasi-linear utilities: for $x \in A$ and payment $p_i \in \mathbb{R}$, $u_i(x, p_i) = v_i(x) - p_i$
- A (direct revelation) mechanism $M = (f, p)$ takes as input a valuation profile $\boldsymbol{v} = (v_1, \dots, v_n)$ and returns an alternative $f(\boldsymbol{v})$ and payments $p(\boldsymbol{v})$, where $p_i(\boldsymbol{v})$ is the payment of player i

VOTING RULES VS. MECHANISMS



Voting Rule



Mechanism with payments

AUCTIONING A SINGLE ITEM

- There is one item and n players with bids $b_1, \ldots, b_n$, the highest bidder gets the item; what should they pay?
- $A = \{a_1, \ldots, a_n\}$, where $i \in N$ gets the item in alternative a_i
- Valuations defined by

$$v_i(x) = \begin{cases} b_i, & x = a_i \\ 0, & x \neq a_i \end{cases}$$

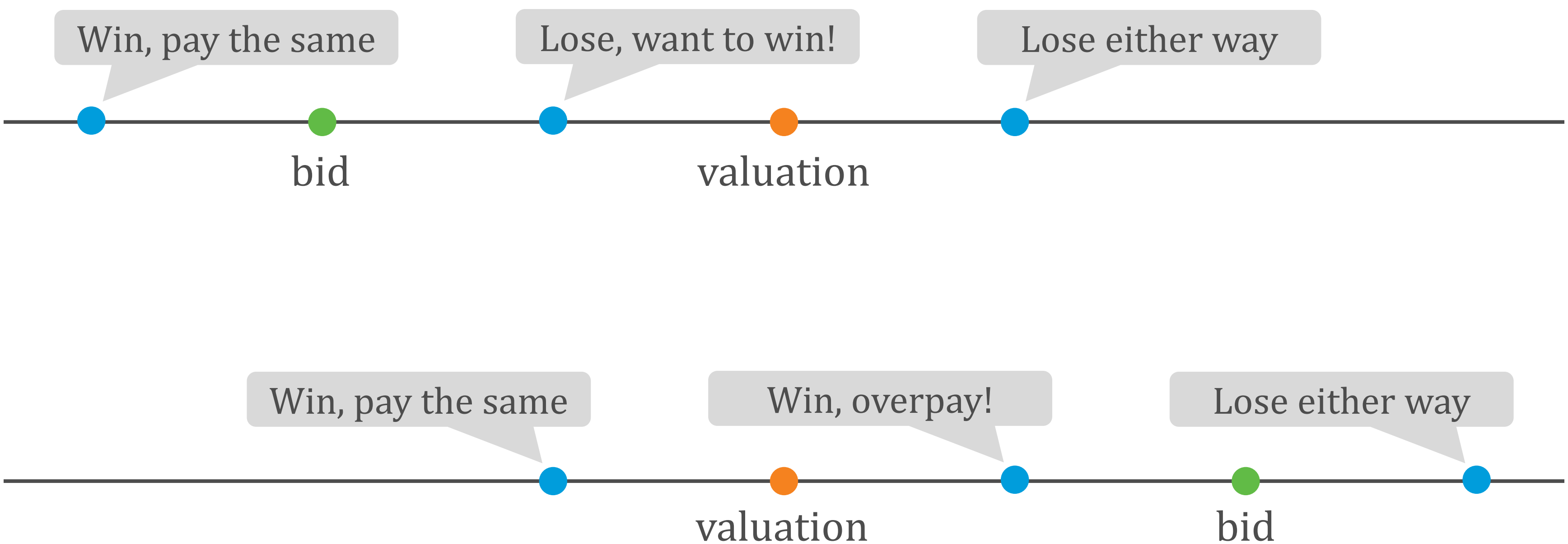
- $f(\boldsymbol{v}) = a_i$ s.t. $b_i = \max_{j \in N} b_j$
- First price: $p_i(\boldsymbol{v}) = b_i$, $p_j(\boldsymbol{v}) = 0$ for $j \neq i$
- Vickrey: $p_i(\boldsymbol{v}) = \max_{j \neq i} b_j$, $p_j(\boldsymbol{v}) = 0$ for $j \neq i$

STRATEGYPROOFNESS

- A mechanism $M = (f, p)$ is strategyproof if for all valuation profiles $\boldsymbol{v}$, for all $i \in N$ and for all v_i',
  $u_i\big(f(\boldsymbol{v}), p_i(\boldsymbol{v})\big) \geq u_i\big(f(v_i', \boldsymbol{v}_{-i}), p_i(v_i', \boldsymbol{v}_{-i})\big)$
- First-price auction is not strategyproof
- Theorem: The Vickrey auction is strategyproof

VICKREY IS STRATEGYPROOF

Cases depend on highest other bid (in blue)



William Vickrey

1914–1996

Professor of economics at Columbia. Also known for receiving the Nobel Prize posthumously.

THE VCG MECHANISM

The Vickrey-Clarke-Groves (VCG) Mechanism is defined by:

- A welfare-maximizing choice rule,

$$f(\boldsymbol{v}) \in \operatorname{argmax}_{x \in A} \sum_{i \in N} v_i(x)$$

- A payment rule p, where A^{-i} is the set of alternatives that are available when i is not present, and

$$p_i(\boldsymbol{v}) = \max_{x \in A^{-i}} \sum_{j \neq i} v_j(x) - \sum_{j \neq i} v_j\big(f(\boldsymbol{v})\big)$$

THE VCG MECHANISM: EXAMPLE

Consider an auction with one item, player 1 has value $7 for getting the item and player 2 has value $3

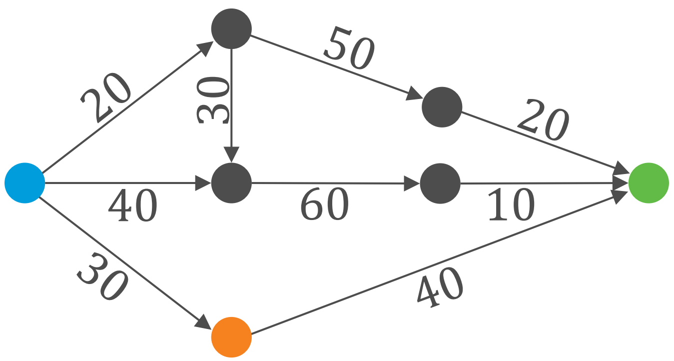
Poll 1

Under VCG, player 1 gets the item and pays:

- $3
- $4
- $5
- $6

THE VCG MECHANISM: EXAMPLE

Alternatives are paths from blue to green, players are edges, each with a cost. Value of an edge is minus its cost if used. A^{-i} are paths that don't include the edge associated with player i.



Poll 2

Under VCG, the blue-orange edge is paid:

- 0
- 50
- 60
- 90

VCG IS STRATEGYPROOF

- **Theorem:** VCG is strategyproof
- **Proof:** Recall $f(\boldsymbol{v}) \in \text{argmax}_{x \in A} \sum_{j \in N} v_j(x)$, the utility of player i is

$$v_i\big(f(\boldsymbol{v})\big) - \left[\max_{x' \in A^{-i}} \sum_{j \neq i} v_j(x') - \sum_{j \neq i} v_j\big(f(\boldsymbol{v})\big)\right]$$

$$= \underbrace{\sum_{j \in N} v_j\big(f(\boldsymbol{v})\big)}_{\text{Maximized at } f(\boldsymbol{v})} - \underbrace{\max_{x' \in A^{-i}} \sum_{j \neq i} v_j(x')}_{\text{Independent of } v_i}$$

VCG IS INDIVIDUALLY RATIONAL

- For a mechanism $M = (f, p)$, denote by $f(\boldsymbol{v}_{-i}) \in A^{-i}$ the outcome of the mechanism when i isn't present
- M is individually rational if for any valuation profile $\boldsymbol{v}$ and any $i \in N$, $u_i\big(f(\boldsymbol{v}), p_i(\boldsymbol{v})\big) \geq u_i(f(\boldsymbol{v}_{-i}), 0)$
- Theorem: VCG is individually rational
- Proof: The difference $u_i\big(f(\boldsymbol{v}), p_i(\boldsymbol{v})\big) - u_i(f(\boldsymbol{v}_{-i}), 0)$ is

$$v_i\big(f(\boldsymbol{v})\big) - \left[\sum_{j \neq i} v_j\big(f(\boldsymbol{v}_{-i})\big) - \sum_{j \neq i} v_j\big(f(\boldsymbol{v})\big)\right] - v_i\big(f(\boldsymbol{v}_{-i})\big)$$

$$= \max_{x \in A} \sum_{j \in N} v_j(x) - \sum_{j \in N} v_j\big(f(\boldsymbol{v}_{-i})\big) \geq 0 \ \blacksquare$$