ECONOMICS & COMPUTATION

Spring 2025 | Lecture 4

The Price of Anarchy

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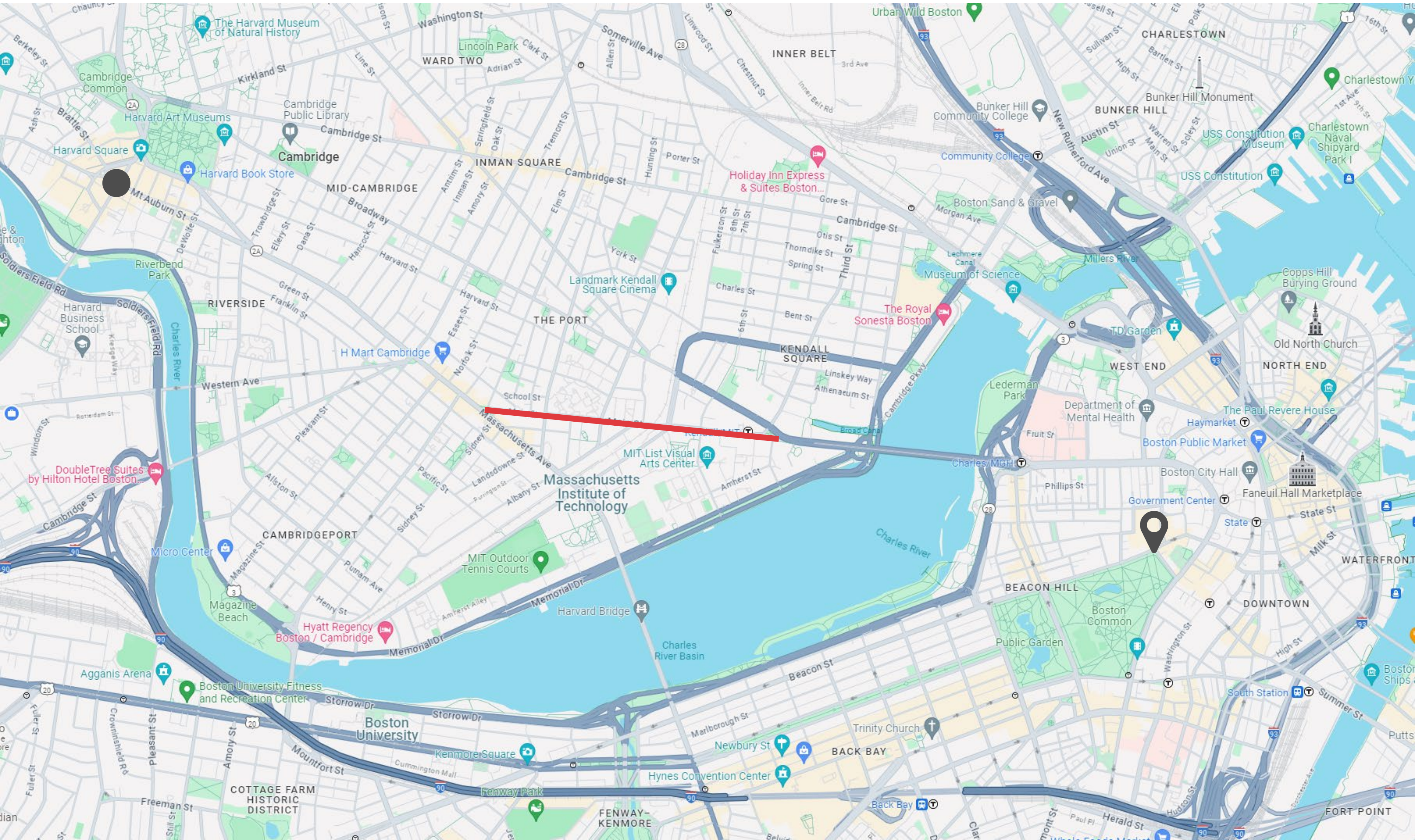
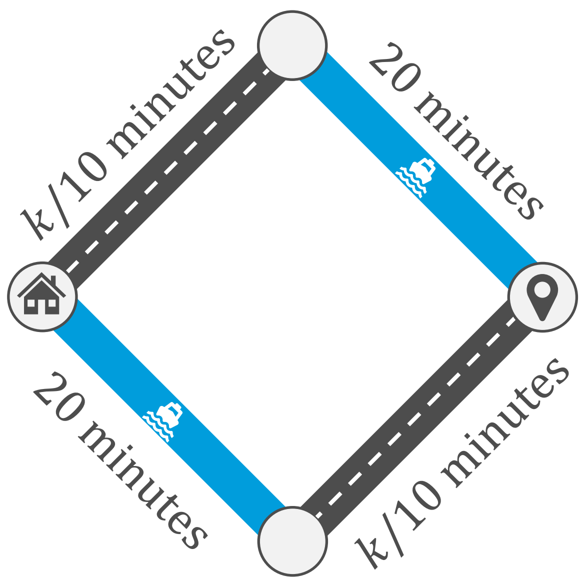
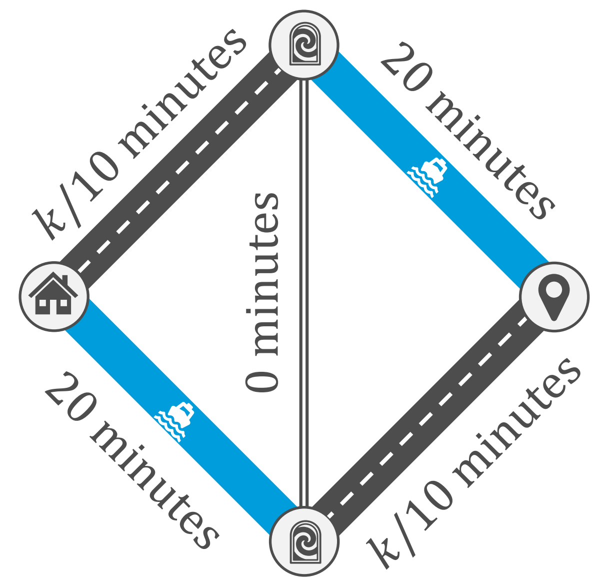
CLOSING ROADS SPEEDS UP TRAFFIC



BRAESS' PARADOX

200 travelers want to get from home to work





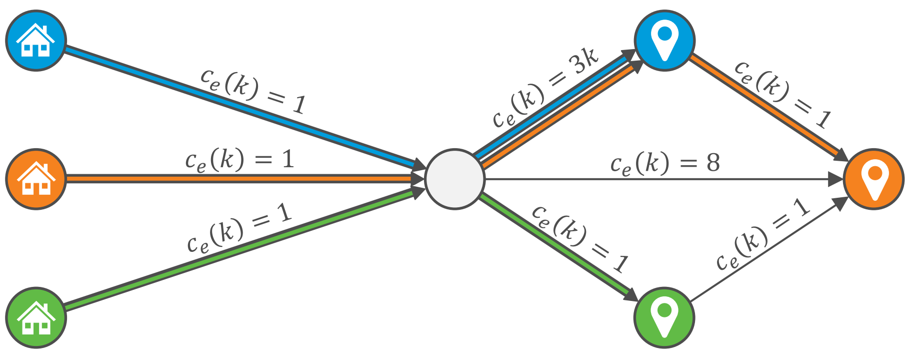
In Nash equilibrium, 100 travelers take each route and the travel time is 30 minutes

In Nash equilibrium, all travelers take the zigzag route and the travel time is 40 minutes

ROUTING GAMES

- An (atomic) routing game consists of:
  - A set of players $N = \{1, \dots, n\}$
  - A directed graph $G = (V, E)$
  - For each $e \in E$, a nonnegative and nondecreasing cost function $c_e: \mathbb{N} \to \mathbb{R}^+$
  - For each $i \in N$, a source and sink $a_i, b_i \in V$
- The strategy set S_i of each i is $a_i \to b_i$ paths
- In a strategy profile $\boldsymbol{s}$, $n_e(\boldsymbol{s})$ players are using edge e
- The cost of player i is $\text{cost}_i(\boldsymbol{s}) = \sum_{e \in s_i} c_e\big(n_e(\boldsymbol{s})\big)$
- The social cost is $\text{cost}(\boldsymbol{s}) = \sum_{i \in N} \text{cost}_i(\boldsymbol{s})$

ROUTING GAMES: EXAMPLE



The orange path has a cost of 8 and the social cost is 17. The optimal solution has social cost 9.

POTENTIAL GAMES

- We were talking about pure Nash equilibria, but how do we know they even exist?
- A game is an exact potential game if there exists a function $\Phi: \prod_{i=1}^{n} S_i \to \mathbb{R}$ such that for all $i \in N$, for all $\boldsymbol{s} \in \prod_{i=1}^{n} S_i$, and for all $s_i' \in S_i$,
$\text{cost}_i(s_i', \boldsymbol{s}_{-i}) - \text{cost}_i(\boldsymbol{s}) = \Phi(s_i', \boldsymbol{s}_{-i}) - \Phi(\boldsymbol{s})$
- The existence of an exact potential function implies the existence of a pure Nash equilibrium — why?

ROUTING GAMES HAVE POTENTIAL

- Theorem: Routing games are exact potential games
- Proof:
  - Define the potential function

$$\Phi(\boldsymbol{s}) = \sum_{e \in E} \sum_{k=1}^{n_e(\boldsymbol{s})} c_e(k)$$

  - Suppose player i deviates from path s_i to path s_i', then $\text{cost}_i(s_i', \boldsymbol{s}_{-i}) - \text{cost}_i(\boldsymbol{s})$ is

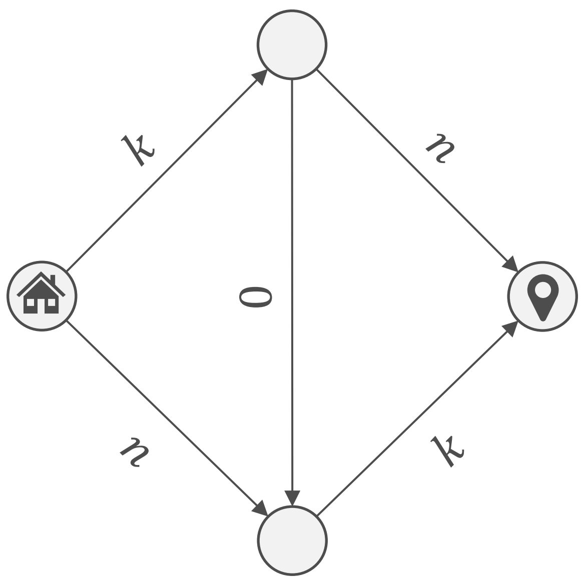
$$\sum_{e \in s_i' \setminus s_i} c_e(n_e(\boldsymbol{s}) + 1) - \sum_{e \in s_i \setminus s_i'} c_e\big(n_e(\boldsymbol{s})\big)$$

  - This is precisely $\Phi(s_i', \boldsymbol{s}_{-i}) - \Phi(\boldsymbol{s})$ ■

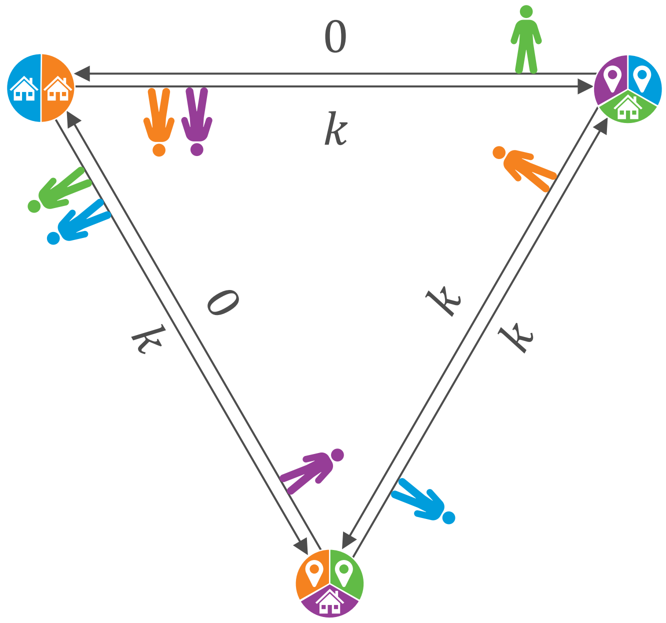
PRICE OF ANARCHY: DEFINITION

- Fix a class of games, an objective function, and an equilibrium concept
- The price of anarchy is the worst-case ratio between the worst objective function value of an equilibrium of the game, and that of the optimal solution
- In this lecture:
    - Objective function = social cost
    - Equilibrium concept = pure Nash equilibrium

POA OF ROUTING GAMES



Lower bound: 4/3



Lower bound: 5/2

POA OF ROUTING GAMES

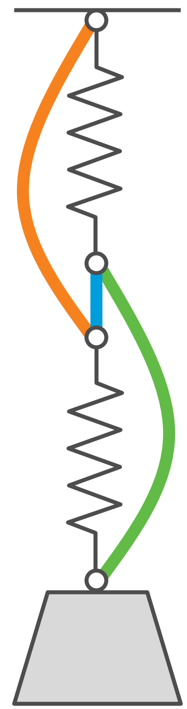
- **Theorem:** For any routing game with linear cost functions, the price of anarchy is at most 2.5
- All in all:
  - $\forall$routing games with linear cost functions, $\forall$NE $\boldsymbol{s}$, $\text{cost}(\boldsymbol{s}) \leq 2.5 \cdot \text{cost}(\text{OPT})$
  - $\exists$routing games with linear cost functions, $\exists$NE $\boldsymbol{s}$, $\text{cost}(\boldsymbol{s}) \geq 2.5 \cdot \text{cost}(\text{OPT})$

Poll 1

Suppose there are n players and two edges between a common source and a common sink, one with cost 1 and one with cost $(k/n)^p$. As $p \to \infty$, what lower bound does this imply for PoA?

○ 2.5 ○ 4 ○ n ○ ∞

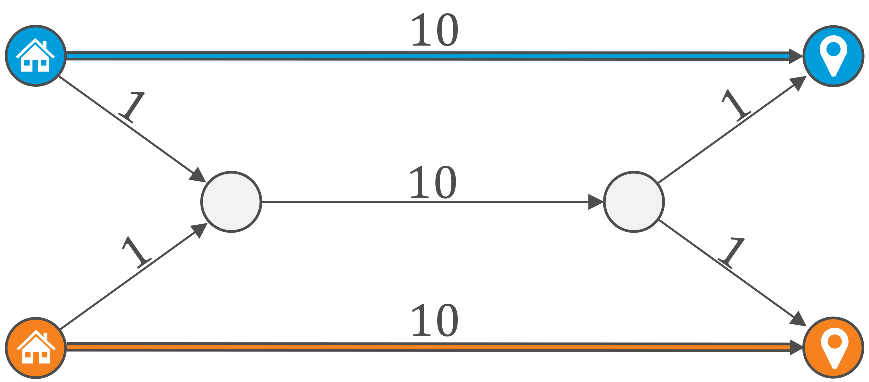
BREAK: THE SPRING PARADOX



What happens when the blue string is cut?
And how is this related to Braess' Paradox?
https://www.youtube.com/watch?v=Cg73j3QYRJc

COST SHARING GAMES

Cost sharing games are the same as routing games with one exception: each edge has a fixed cost c_e that is split among players using it



This is a Nash equilibrium with a social cost of 20, whereas the optimal solution has a social cost of 14

POA OF COST SHARING GAMES

- An example with n players and two edges between a common source and a common sink with costs 1 and n shows the price of anarchy of cost sharing games is at least n — why?
- Theorem: The price of anarchy of cost sharing games is at most n
- Proof:
  - Let $\boldsymbol{s}$ be a Nash equilibrium and let $\boldsymbol{s}^\star$ be an optimal solution
  - For all i, it holds that $\text{cost}_i(\boldsymbol{s}) \leq \text{cost}_i(s_i^\star, \boldsymbol{s}_{-i})$ because i can't gain from unilaterally deviating
  - But $\text{cost}_i(s_i^\star, \boldsymbol{s}_{-i}) \leq n \cdot \text{cost}_i(\boldsymbol{s}^\star)$, because in the worst case i pays for its path alone in the former and splits each edge cost n ways in the latter
  - Summing over i, we get that $\text{cost}(\boldsymbol{s}) \leq n \cdot \text{cost}(\boldsymbol{s}^\star)$ ■

PRICE OF STABILITY: DEFINITION

- Fix a class of games, an objective function, and an equilibrium concept
- The price of stability is the worst-case ratio between the best objective function value of an equilibrium of the game, and that of the optimal solution

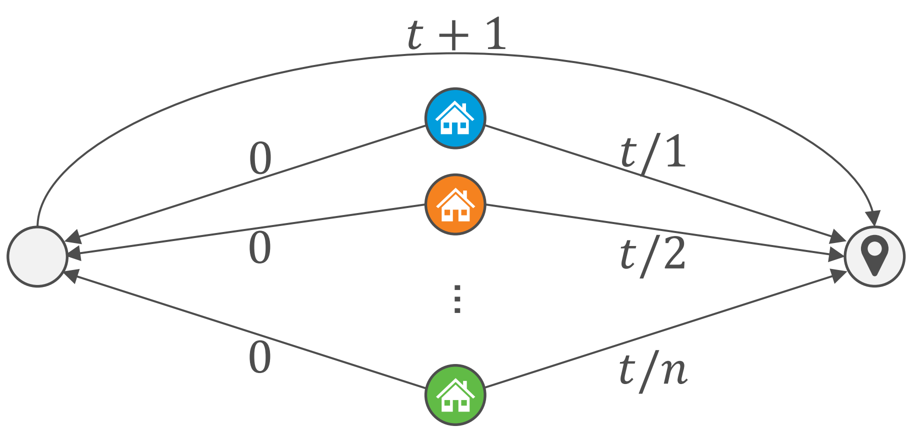
Poll 2

There are n players and two edges between a common source and a common sink, one with cost 1 and one with cost n. What lower bound does this imply on the PoS of cost sharing games?

- ○ 1
- ○ 2
- ○ 3
- ○ n

POS OF COST SHARING GAMES

- Theorem: The price of stability of cost sharing games is at least $H(n) \approx \ln(n)$
- Proof: Consider the following example and let $t \to \infty$ ■



POS OF COST SHARING GAMES

- Like routing games, cost sharing games are exact potential games with essentially the same potential function: $\Phi(\boldsymbol{s}) = \sum_e \sum_{k=1}^{n_e(\boldsymbol{s})} \frac{c_e}{k}$
- Theorem: The price of stability of cost sharing games is at most $H(n)$
- Proof:
  - It holds that $\text{cost}(\boldsymbol{s}) \leq \Phi(\boldsymbol{s}) \leq H(n) \cdot \text{cost}(\boldsymbol{s})$
  - Take a strategy profile $\boldsymbol{s}^\star$ that minimizes Φ
  - $\boldsymbol{s}^\star$ is an NE
  - $\text{cost}(\boldsymbol{s}^\star) \leq \Phi(\boldsymbol{s}^\star) \leq \Phi(\text{OPT}) \leq H(n) \cdot \text{cost}(\text{OPT})$ ■

COST SHARING SUMMARY

- Upper bounds: $\forall$cost sharing game,
  - PoA: $\forall$NE $\boldsymbol{s}$,
    $$\text{cost}(\boldsymbol{s}) \leq n \cdot \text{cost}(\text{OPT})$$
  - PoS: $\exists$NE $\boldsymbol{s}$ s.t.
    $$\text{cost}(\boldsymbol{s}) \leq H(n) \cdot \text{cost}(\text{OPT})$$
- Lower bounds: $\exists$cost sharing game s.t.
  - PoA: $\exists$NE $\boldsymbol{s}$ s.t.
    $$\text{cost}(\boldsymbol{s}) \geq n \cdot \text{cost}(\text{OPT})$$
  - PoS: $\forall$NE $\boldsymbol{s}$,
    $$\text{cost}(\boldsymbol{s}) \geq H(n) \cdot \text{cost}(\text{OPT})$$