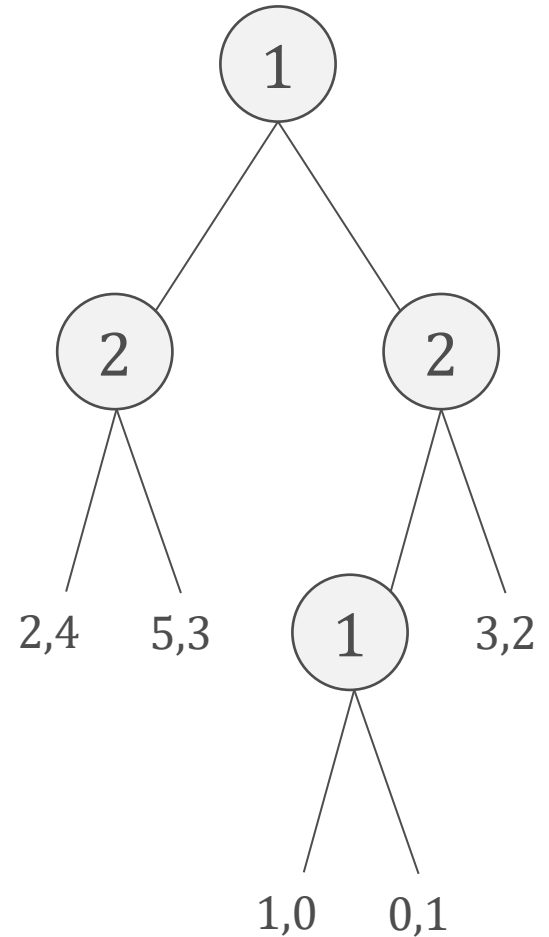Spring 2025 | Lecture 3
Extensive-Form Games
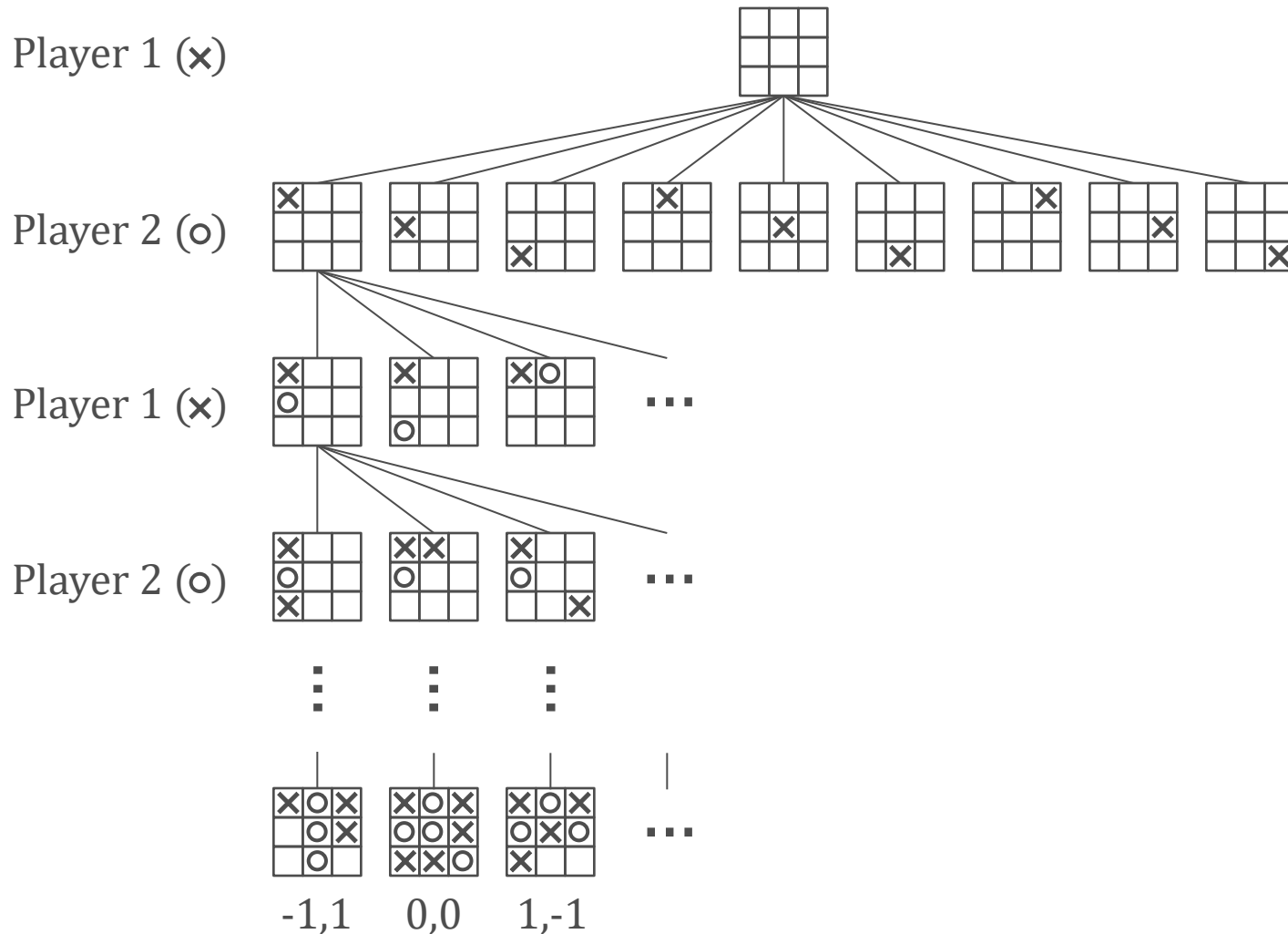Ariel Procaccia | Harvard University

EXTENSIVE-FORM GAMES
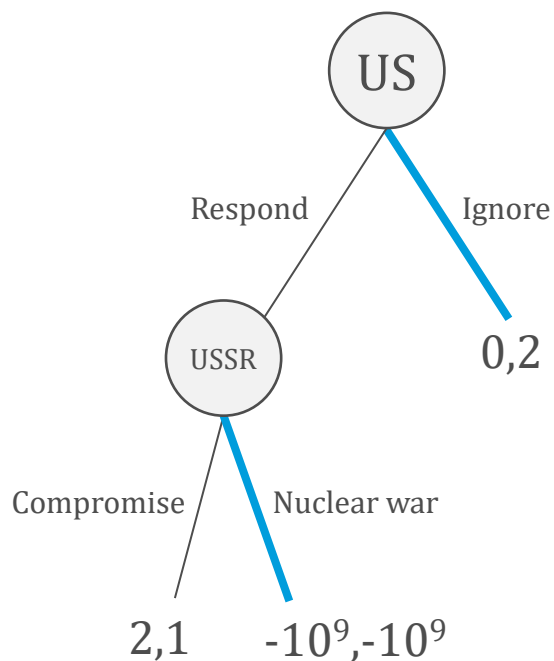
- Moves are done sequentially, not simultaneously
- Game forms a tree
- Nodes are labeled by players
- Leaves show payoffs

EXAMPLE: TIC-TAC-TOE

Player 1 (✖)

Player 2 (○)

Player 1 (✖)   · · ·

Player 2 (○)   · · ·

-1,1   0,0   1,-1

EXTENSIVE VS. NORMAL FORM

	Compromise	Nuclear war
Respond	2,1	$-10^9, -10^9$
Ignore	0,2	0,2

Problem: (ignore, nuclear war) is a Nash equilibrium, but threat isn't credible!

SUBGAME-PERFECT EQUILIBRIUM

- Each subtree forms a subgame

- A set of strategies is a subgame-perfect equilibrium if it is a Nash equilibrium in each subgame

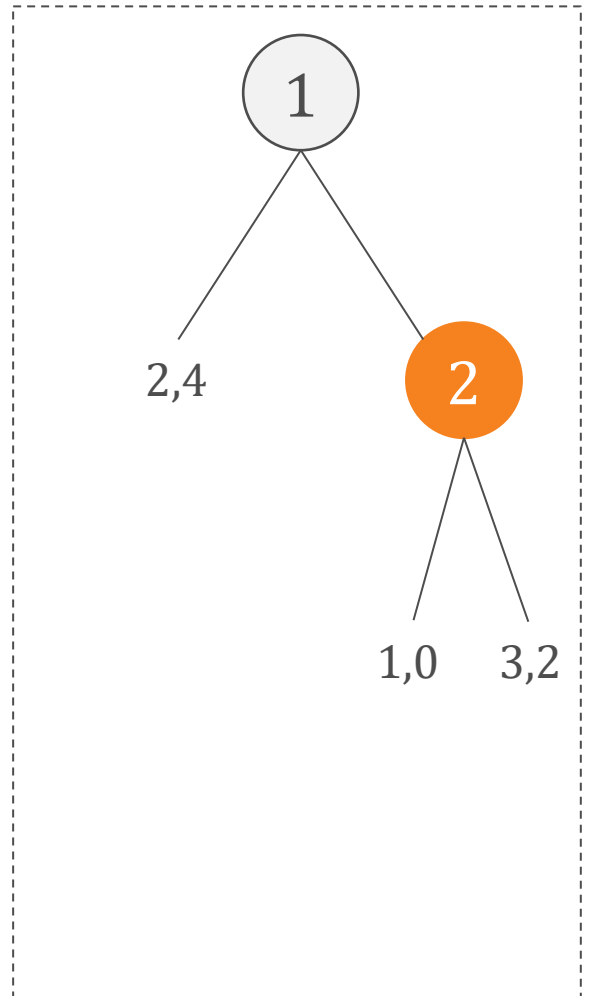- Players may be able to improve their equilibrium payoff by eliminating strategies!
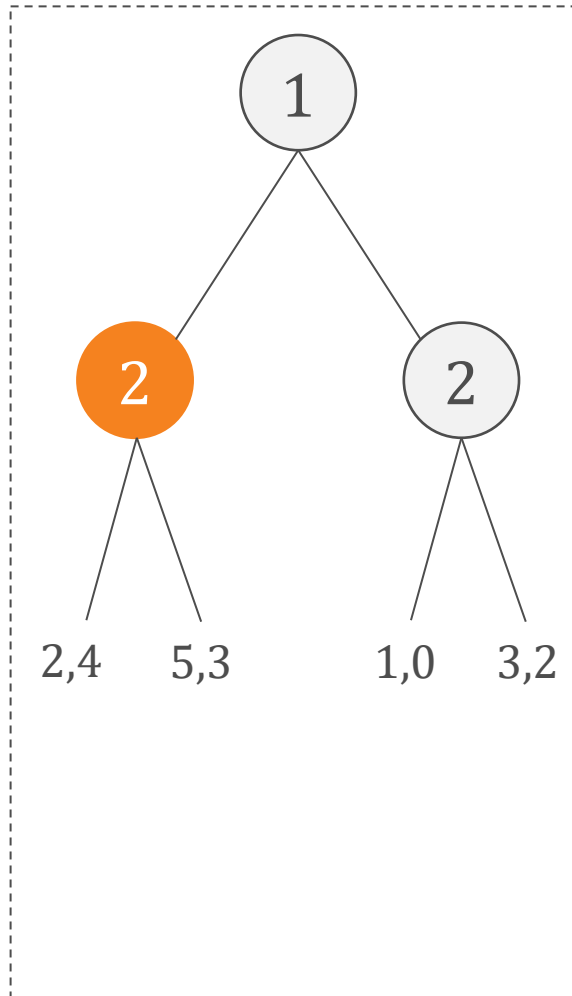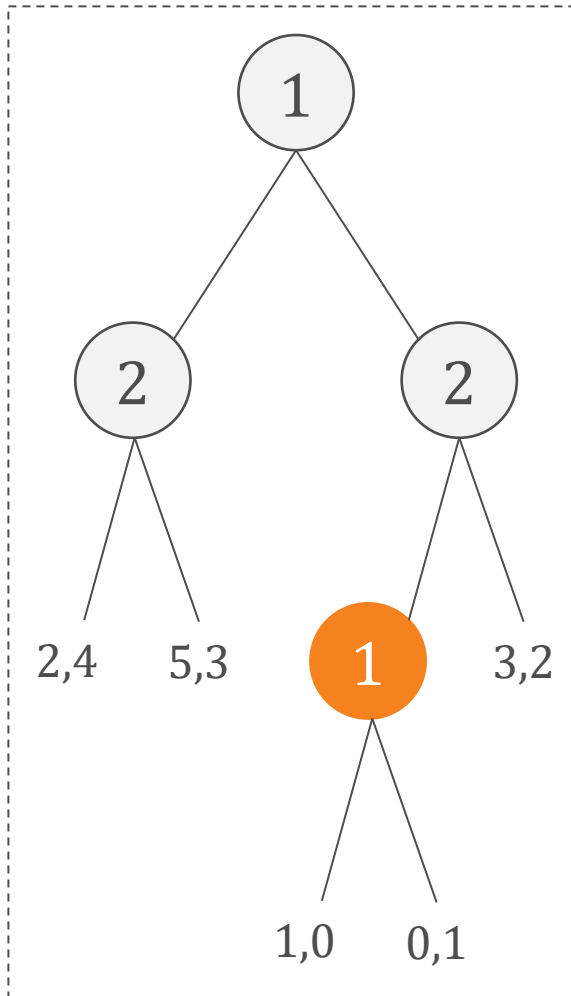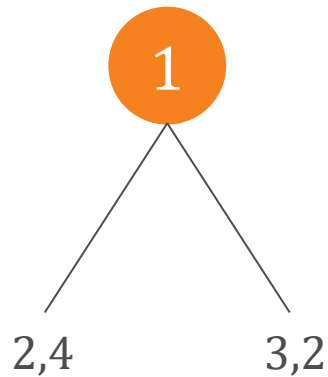
DOOMSDAY MACHINE

https://youtu.be/2yfXgu37iyI
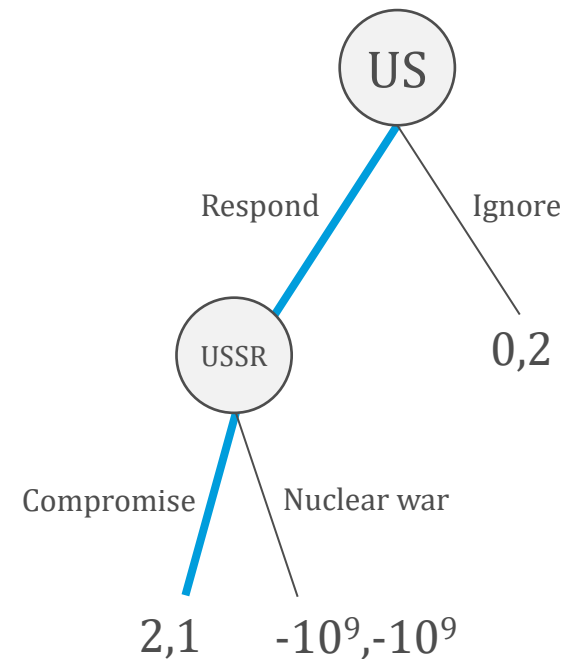
BACKWARD INDUCTION

BACKWARD INDUCTION

1

2,4   3,2

3,2

Subgame-perfect
equilibrium!

1

2   2

2,4   5,3   1   3,2

1,0   0,1
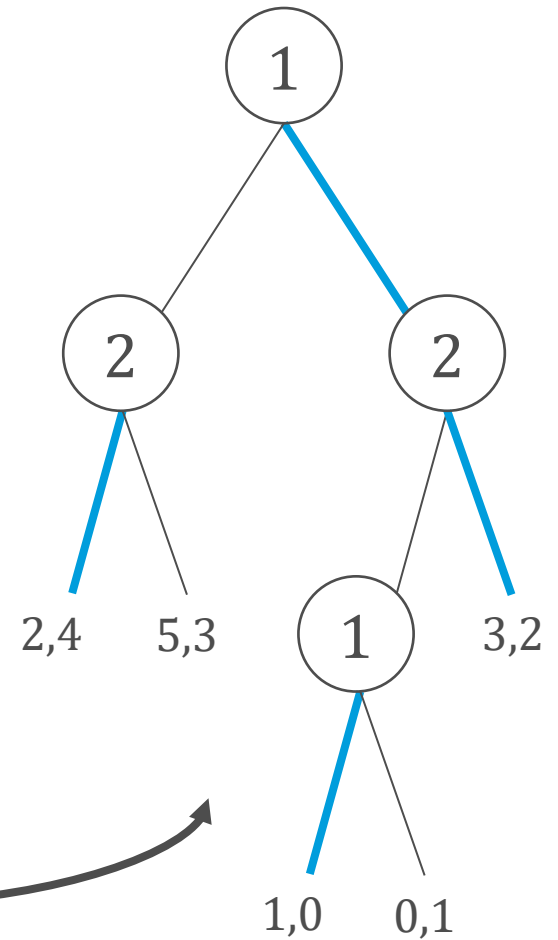
EXAMPLE: CENTIPEDE GAME

| 1 | — | 2 | — | 1 | — | 2 | — | 1 | — | 2 | — 7,5 |

| 1,-1 | 0,2 | 3,1 | 2,4 | 5,3 | 4,6 |
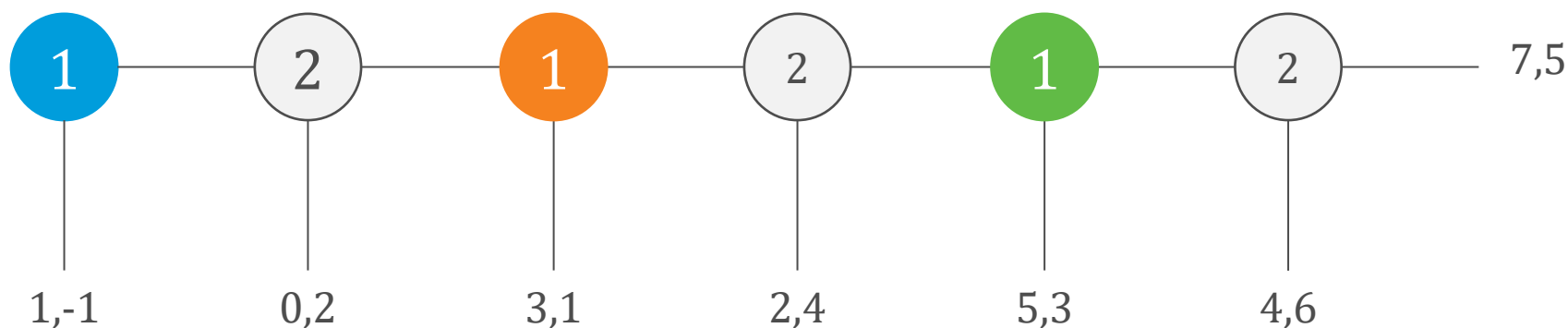
Poll 1

Suppose you are player 1 and you're playing with a random classmate. At which point do you choose down?

○ Blue   ○ Orange   ○ Green   ○ None

IMPERFECT INFORMATION GAMES

- A chance node chooses between several actions according to a known probability distribution
- An information set is a set of nodes that a player may be in, given the available information
- A strategy must be identical for all nodes in an information set
- Reasoning about equilibria becomes much more challenging, so we'll focus on a two-step special case
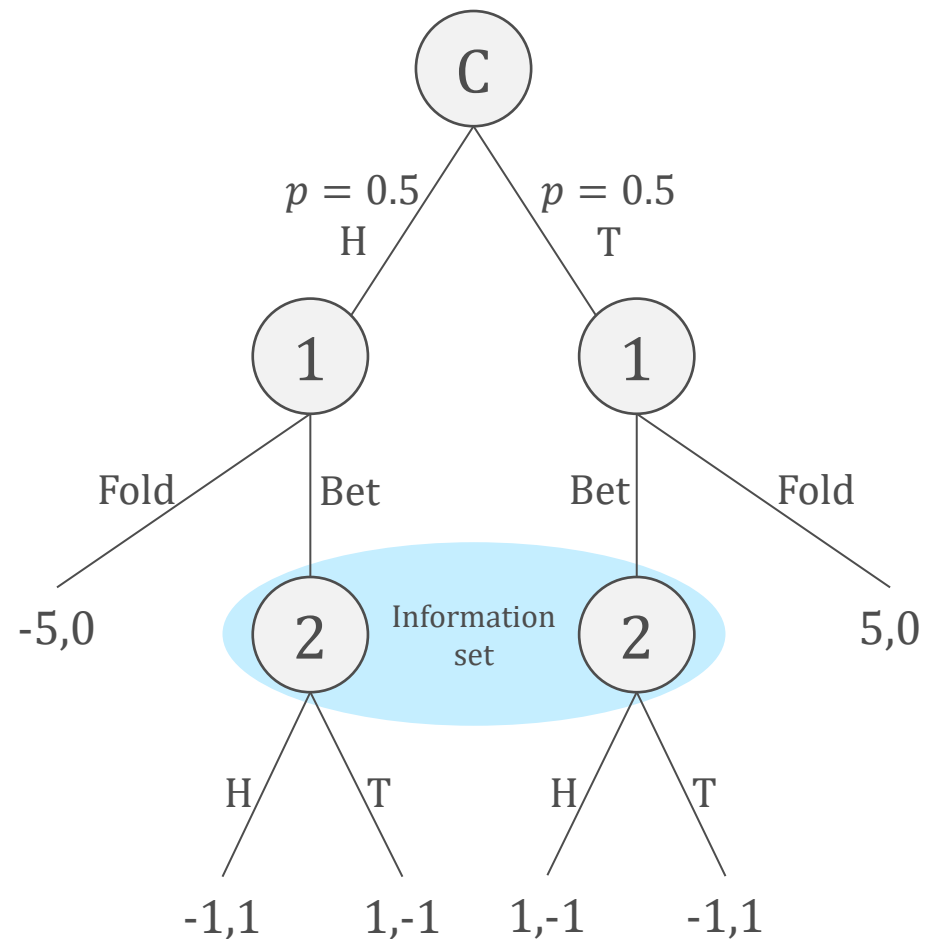
STACKELBERG GAMES

1,1	3,0
0,0	2,1

- Playing up is a dominant strategy for row player
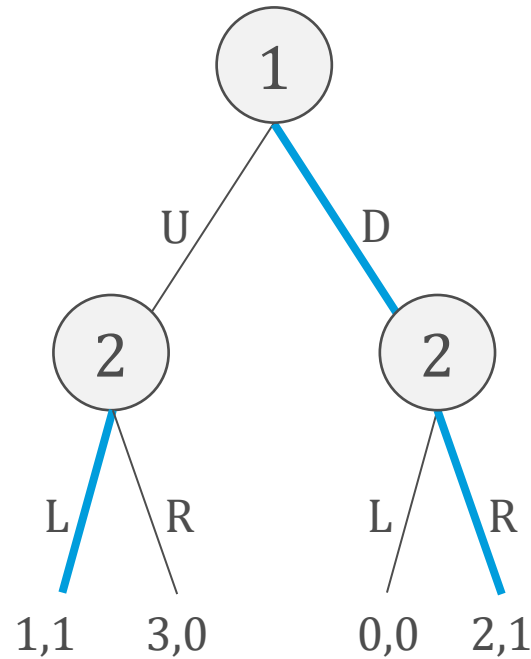- So column player would play left
- Therefore, (1,1) is the only Nash equilibrium outcome

STACKELBERG GAMES

1,1	3,0
0,0	2,1

- A Stackelberg game is played as follows:
  - Row player (the leader) commits to playing a row
  - Column player (the follower) observes the commitment and chooses column
- The leader can commit to playing down!

STACKELBERG GAMES

If the leader announces their commitment, the Stackelberg game can be rewritten as an extensive-form game of perfect information

STACKELBERG GAMES

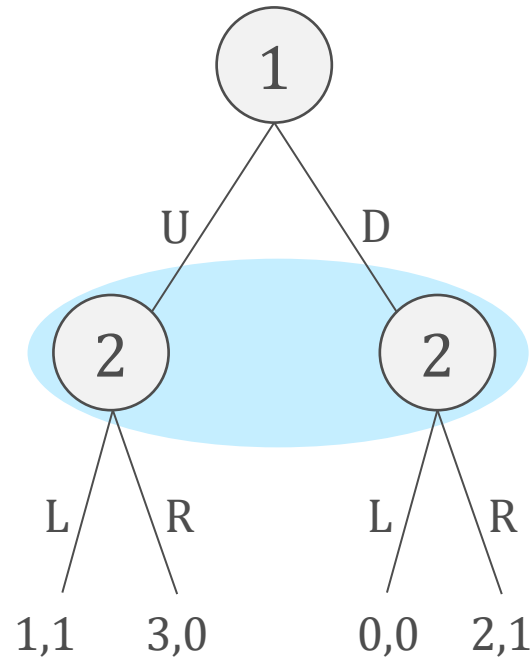|	
1,1	3,0
0,0	2,1

Poll 2

What reward can the leader get by committing to a mixed strategy? (Assume the follower breaks ties in favor of the leader)

○ 1   ○ 1.5   ○ 2   ○ 2.5

STACKELBERG GAMES

Randomness helps the leader due to imperfect information

STACKELBERG EQUILIBRIUM

- For a mixed strategy x_1 of the leader, define the best response set of the follower as
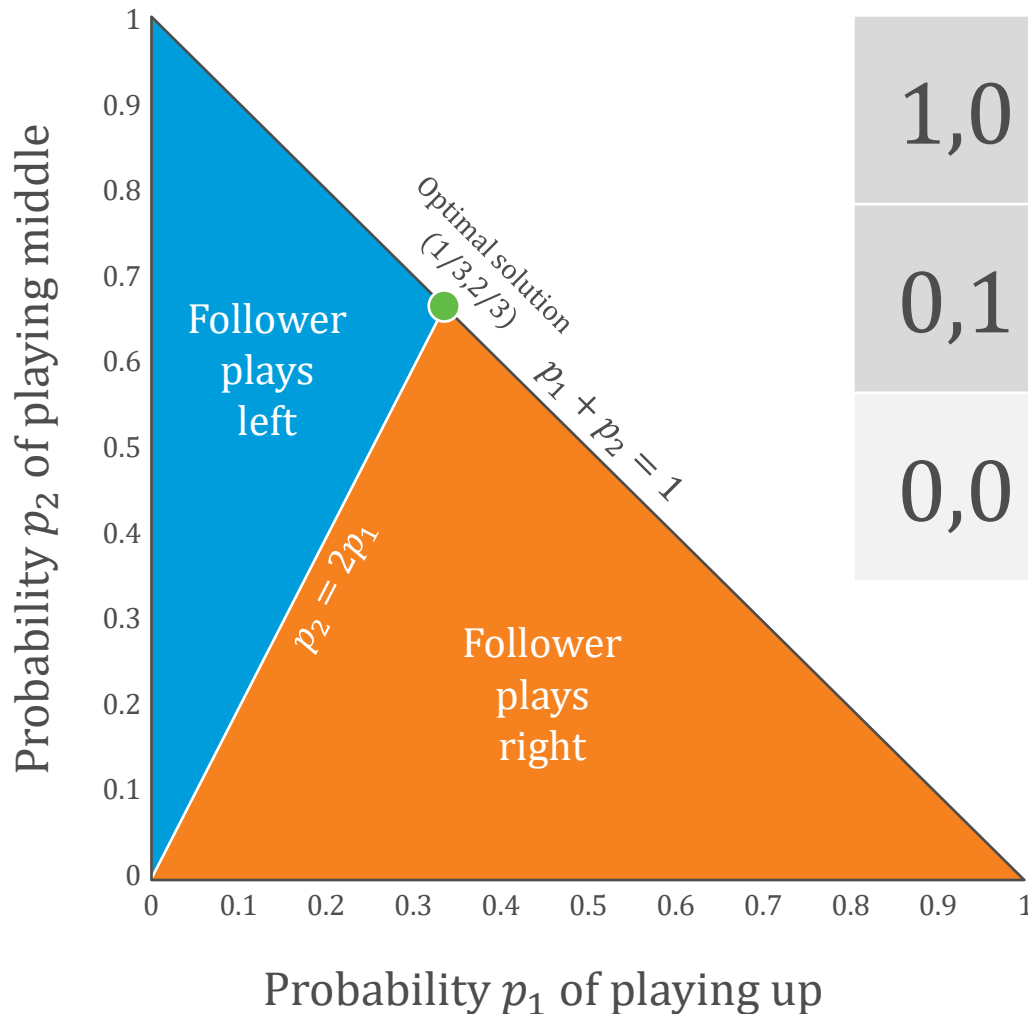$$B_2(x_1) = \text{argmax}_{s_2 \in S} \, u_2(x_1, s_2)$$

- In a strong Stackelberg equilibrium (SSE), the leader plays a mixed strategy in
$$\text{argmax}_{x_1 \in \Delta(S)} \, \max_{s_2 \in B_2(x_1)} \, u_1(x_1, s_2),$$

  where $\Delta(S)$ is the set of mixed strategies

- We'll next see that an SSE can be computed via linear programming
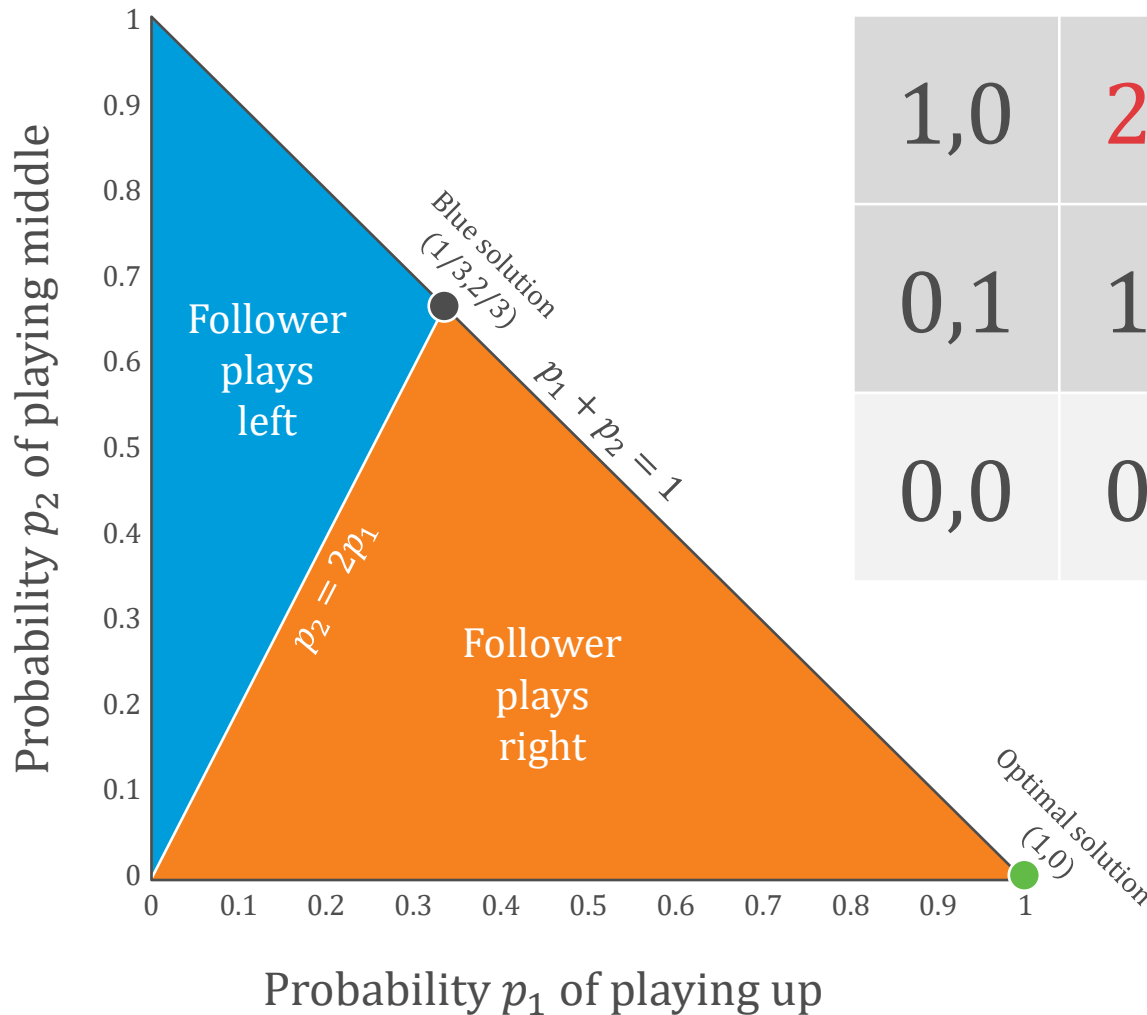
COMPUTING SSE: EXAMPLE

COMPUTING SSE: EXAMPLE

Probability p_2 of playing middle (y-axis)

Probability p_1 of playing up (x-axis)

Follower plays left

Follower plays right

$p_2 = 2p_1$

$p_1 + p_2 = 1$

Blue solution $(1/3, 2/3)$

Optimal solution $(1,0)$

1,0	**2,2**
0,1	1,0
0,0	0,0

max p_1
s.t. $p_2 \geq 2p_1$
    $p_1 + p_2 \leq 1$
    $p_1, p_2 \geq 0$

max $2p_1 + p_2$
s.t. $p_2 \leq 2p_1$
    $p_1 + p_2 \leq 1$
    $p_1, p_2 \geq 0$

COMPUTING SSE: ALGORITHM

- The leader's mixed strategy is defined by variables $x(s_1)$, which give the probability of playing each strategy $s_1 \in S$
- For each follower strategy $s_2^\star$, we compute a strategy x for the leader such that
  - Playing $s_2^\star$ is a best response for the follower
  - Under this constraint, x is optimal

$$\max \sum_{s_1 \in S} x(s_1) u_1(s_1, s_2^\star)$$

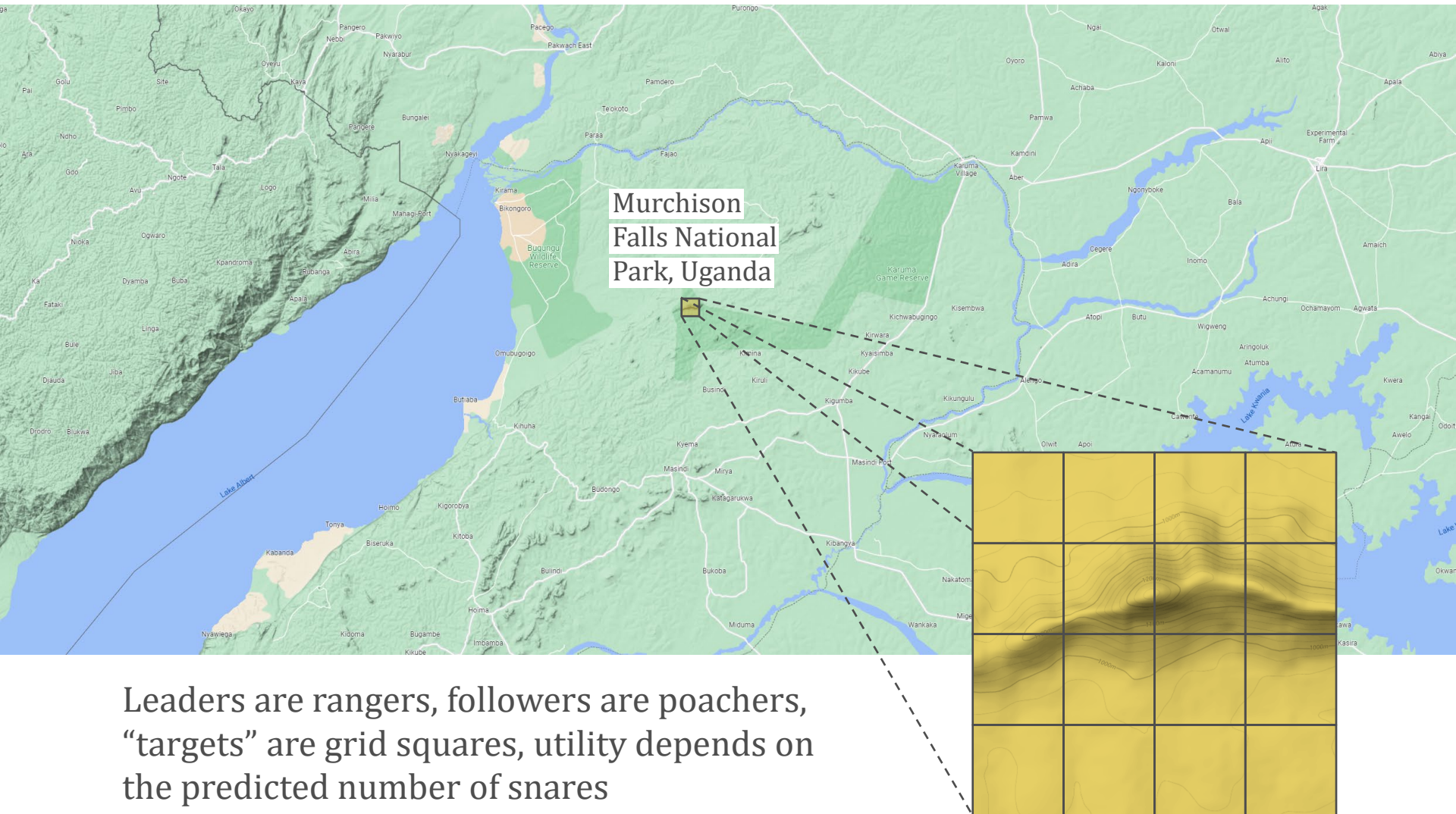$$\text{s.t} \quad \forall s_2 \in S, \sum_{s_1 \in S} x(s_1) u_2(s_1, s_2^\star) \geq \sum_{s_1 \in S} x(s_1) u_2(s_1, s_2)$$

$$\sum_{s_1 \in S} x(s_1) = 1$$
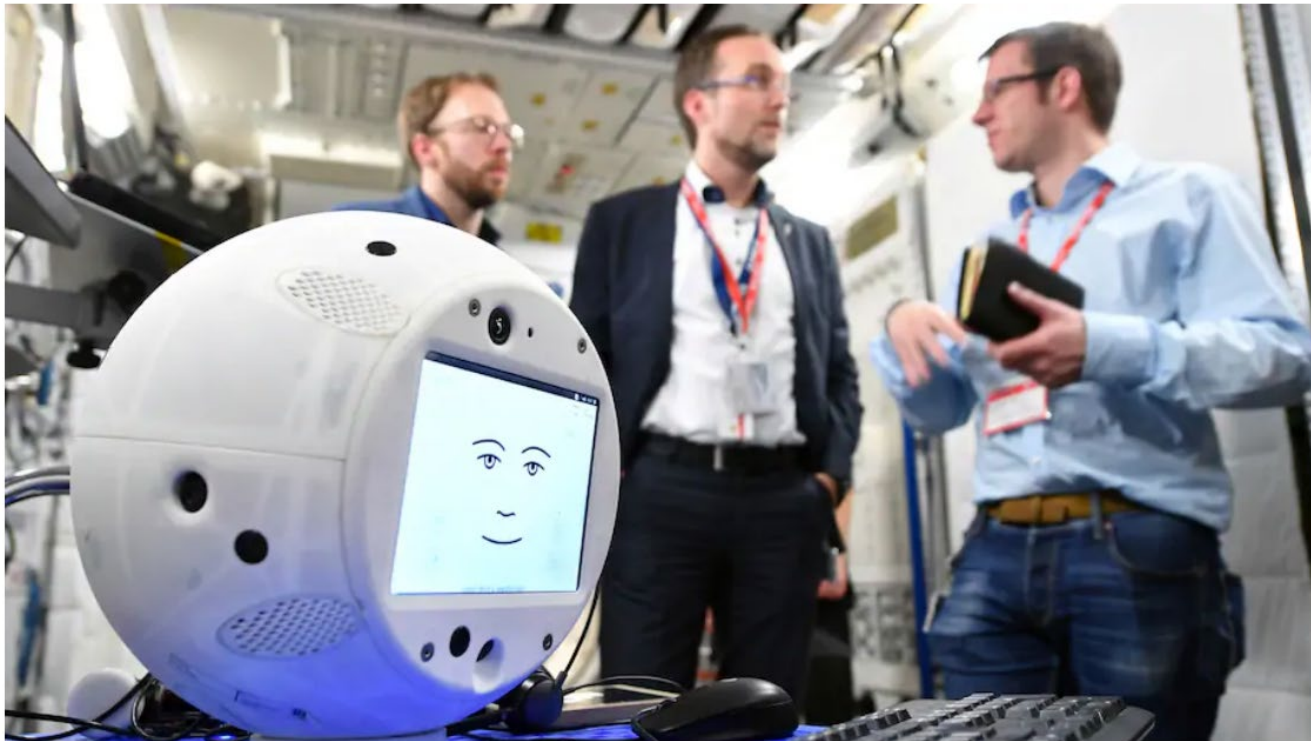
- Take the x resulting from the "best" $s_2^\star$

SECURITY GAMES

Milind
Tambe

WILDLIFE PROTECTION

Murchison Falls National Park, Uganda

Leaders are rangers, followers are poachers, "targets" are grid squares, utility depends on the predicted number of snares

It's time for AI to outgrow gaming

The Cimon (Crew Interactive MObile companioN) robot is shown during a communications test at the ESA European Astronaut Center in Cologne-Porz, Germany, on Jan. 30, 2018. (T. Bourry/AP)

Opinion by **Ariel Procaccia**

March 20, 2020 at 8:52 a.m. EDT