ECONOMICS & COMPUTATION

Spring 2025 | Lecture 17

Cascade Models

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MOTIVATION

- Spread of ideas and new behaviors through a population
- Examples:
  - Political movements
  - Adoption of technological innovations
  - Success of new product
- Process starts with early adopters and spreads through the social network

COORDINATION GAMES

- Undirected, connected graph $G = (V, E)$
- $V = \{1, \dots, n\}$ is the set of players
- Each $i \in V$ chooses an action $a_i \in \{0,1\}$, and $\boldsymbol{a} = (a_1, \dots, a_n)$ is the action profile
- Player i has neighborhood N_i and degree $d_i = |N_i|$
- $n_{i,b}(\boldsymbol{a}_{-i})$ denotes the number of players in N_i playing action b
- For $q \in [0,1]$, the utility of player i is

$$u_i(\boldsymbol{a}) = \begin{cases} (1-q) \cdot n_{i,1}\,(\boldsymbol{a}_{-i}), & a_i = 1 \\ q \cdot n_{i,0}\,(\boldsymbol{a}_{-i}), & a_i = 0 \end{cases}$$

COORDINATION GAMES

- Let us first consider simultaneous-move coordination games
- The best response of player i is 1 if and only if at least a q-fraction of N_i play 1:

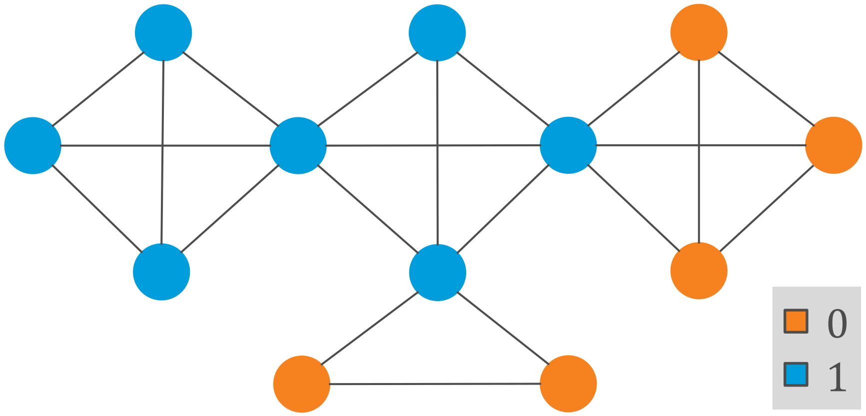
$$0 \leq (1-q)n_{i,1}(\boldsymbol{a}_{-i}) - qn_{i,0}(\boldsymbol{a}_{-i})$$
$$= n_{i,1}(\boldsymbol{a}_{-i}) - q \cdot d_i$$

Poll 1

How many pure Nash equilibria are guaranteed to exist in a coordination game?

○ 0 ○ 1 ○ 2 ○ n

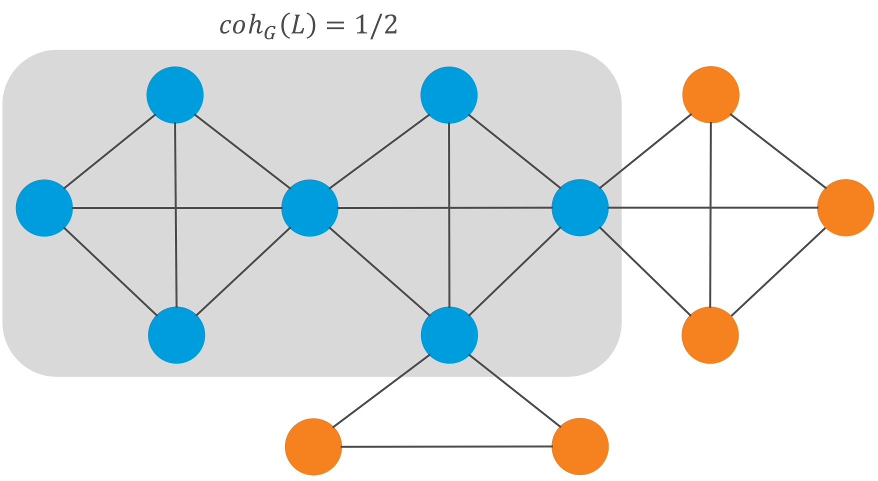
COORDINATION GAMES: EXAMPLE



Nash equilibrium for $q = 1/2$

COHESIVENESS

- The cohesiveness of a set $L \subseteq V$ in G is
$$\mathrm{coh}_G(L) = \min_{i \in L} \left(\frac{|N_i \cap L|}{|N_i|} \right)$$
- We adopt the convention that $\mathrm{coh}_G(\emptyset) = 1$



COHESIVENESS

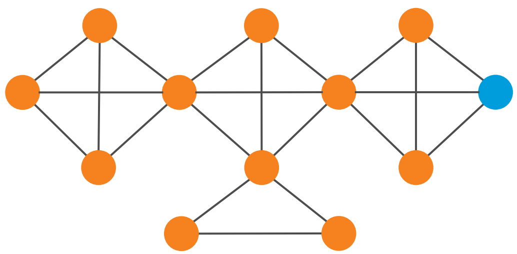
- Theorem: Action profile $\boldsymbol{a}$ is a pure-strategy Nash equilibrium if and only if the sets $X_0 = \{i \in V : a_i = 0\}$ and $X_1 = \{i \in V : a_i = 1\}$ satisfy $\text{coh}_G(X_0) \geq 1 - q$ and $\text{coh}_G(X_1) \geq q$
- Proof:
  - The players in X_1 play a best response if and only if for each of them, the fraction of neighbors playing 1 is at least q, which is equivalent to $|N_i \cap X_1|/|N_i| \geq q$
  - A symmetric argument holds for X_0 ■

A CASCADE MODEL

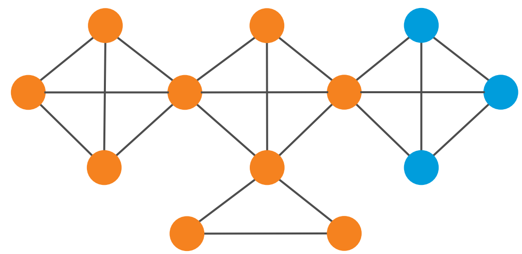
- A set of seeds initially adopt 1, others play 0
- The process is progressive, in the sense that agents only switch from 0 to 1
- We say that a player playing 1 is active and a player switching from 0 to 1 is activated
- In each round, each inactive player with at least a q-fraction of active neighbors is activated

CASCADE MODEL: EXAMPLE

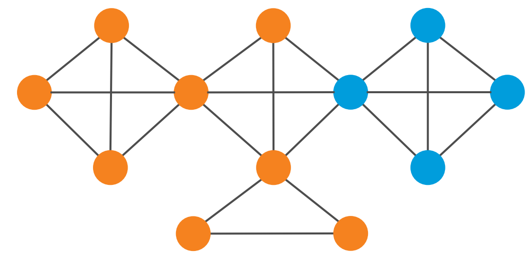
$$q = 1/3$$



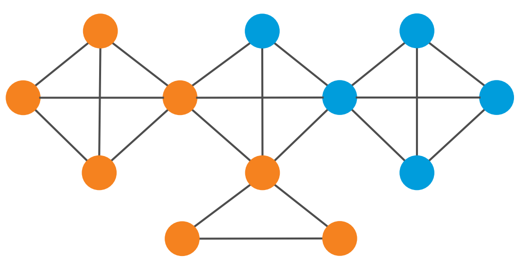
Round 1



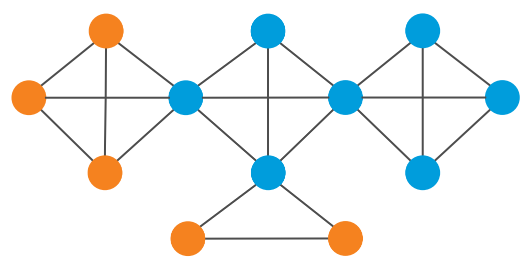
Round 2



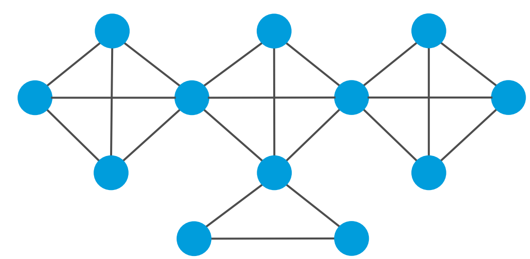
Round 3



Round 4



Round 5

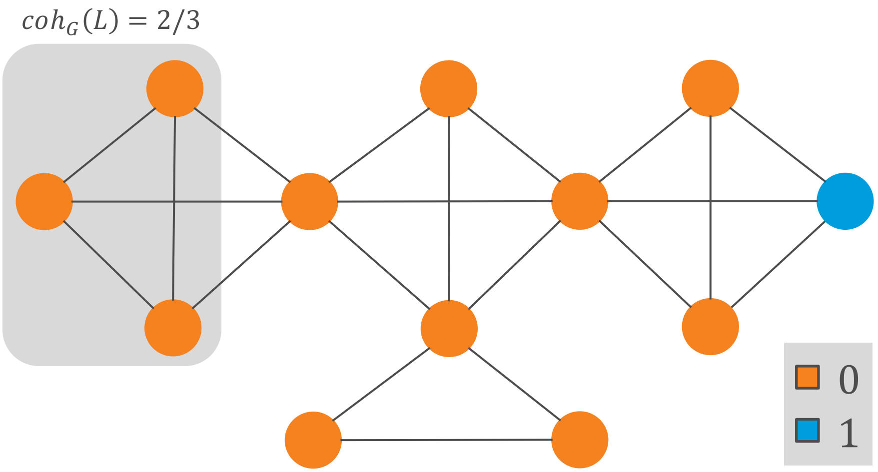


Round 6

CONTAGION

- A set of seeds is contagious if all vertices become activated
- Theorem: Seed set S is contagious if and only if $\mathrm{coh}_G(L) \leq 1 - q$ for every $L \subseteq V \setminus S$
- Proof:
  - Suppose $\mathrm{coh}_G(L) \leq 1 - q$ for every $L \subseteq V \setminus S$, then at any point in the process the set of inactive players $X_0 \subseteq V \setminus S$ has a player with at most a $(1 - q)$ fraction of inactive neighbors, so the process will continue
  - Suppose there is $L \subseteq V \setminus S$ such that $\mathrm{coh}_G(L) > 1 - q$, then no player in L will be the first to be activated ■

CONTAGION: EXAMPLE

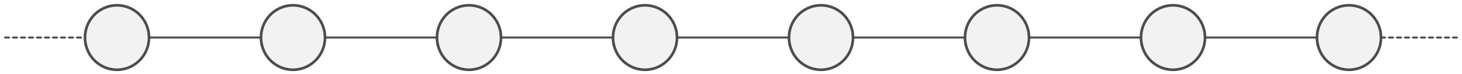


With this S, contagion occurs iff $q \leq 1/3$

INFINITE GRAPHS

- Now assume V is countably infinite and each d_i is bounded
- Easier to be contagious when q is small
- Contagion threshold of $G = \max q$ s.t. $\exists$finite contagious set

INFINITE GRAPHS: EXAMPLE

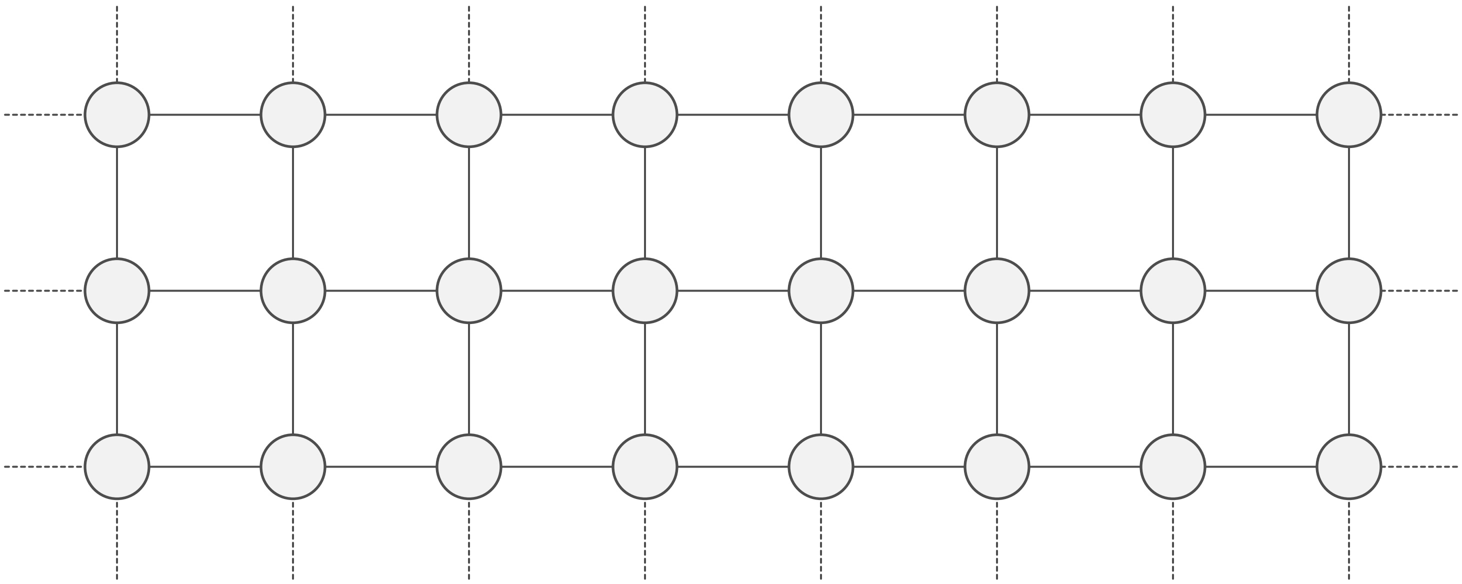


Poll 2

What is the contagion threshold of G?

- 0
- 1/3
- 1/4
- 1/2

INFINITE GRAPHS: EXAMPLE



Poll 3

What is the contagion threshold of G?

- 0
- 1/4
- 1/3
- 1/2

CONTAGION THRESHOLD

- We Saw a graph with contagion threshold 1/2
- Does there exist a graph with contagion threshold $> 1/2$?
- Theorem: The contagion threshold of any graph G is at most 1/2

PROOF OF THEOREM

- Let $q > 1/2$, finite S
- Let S_j be the active nodes at round j
- $\delta(X)$ = set of edges with exactly one end in X
- If $S_{j-1} \neq S_j$ then $|\delta(S_j)| < |\delta(S_{j-1})|$
  - For each $i \in S_j \setminus S_{j-1}$, its edges into S_{j-1} are in $\delta(S_{j-1}) \setminus \delta(S_j)$, and its edges into $V \setminus S_j$ are in $\delta(S_j) \setminus \delta(S_{j-1})$
  - More of the former than the latter because i activated and $q > 1/2$
- $\delta(S)$ is finite and $\delta(S_j) \geq 0$ for all j ■