ECONOMICS & COMPUTATION

Spring 2025 | Lecture 15

Stable Matching

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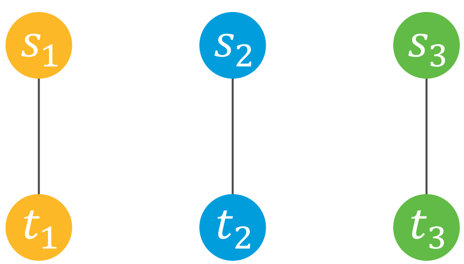
STABLE MATCHINGS

- Match teaching assistants ("students") $S = \{s_1, \dots, s_n\}$ with courses $T = \{t_1, \dots, t_n\}$
- $\pi: S \cup T \to S \cup T$ is a matching such that for all $s \in S$ and $t \in T$, $\pi(s) = t \Leftrightarrow \pi(t) = s$
- Each $s \in S$ has a ranking σ_s over T, and each $t \in T$ has a ranking σ_t over S
- A blocking pair for π is $(s, t) \in S \times T$ such that $s \succ_{\sigma_t} \pi(t)$ and $t \succ_{\sigma_s} \pi(s)$
- A matching π if stable if there is no blocking pair

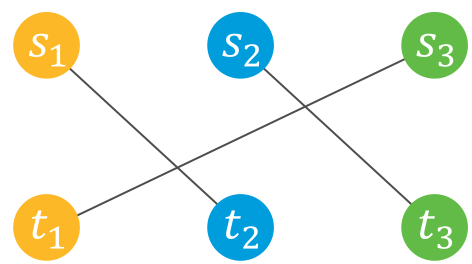
STABLE MATCHING: EXAMPLE

s_1	s_2	s_3
t_2	t_1	t_1
t_1	t_3	t_2
t_3	t_2	t_3

t_1	t_2	t_3
s_1	s_3	s_1
s_3	s_1	s_3
s_2	s_2	s_2



Unstable: (s_1, t_2) blocks



Stable

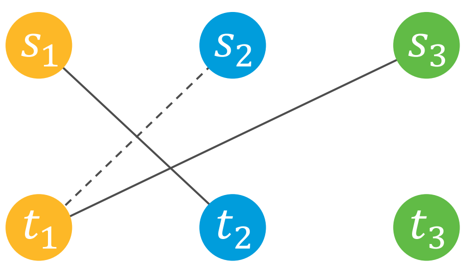
DEFERRED ACCEPTANCE

- In the student-proposing deferred acceptance algorithm, each course is initially unmatched
- In Round 1:
  - Each student s makes a proposal to their most preferred course
  - Each course t that has received a proposal tentatively accepts the most preferred student from those who have proposed and permanently rejects other proposals
- In subsequent rounds:
  - Each student s whose proposal was rejected in the previous round makes a proposal to their next most preferred course
  - Each course t with a new proposal tentatively accepts the most preferred student from their current offers and permanently rejects other proposals

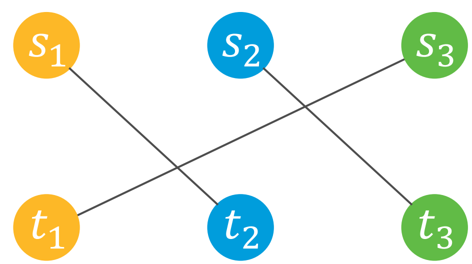
DEFERRED ACCEPTANCE: EXAMPLE

s_1	s_2	s_3
t_2	t_1	t_1
t_1	t_3	t_2
t_3	t_2	t_3

t_1	t_2	t_3
s_1	s_3	s_1
s_3	s_1	s_3
s_2	s_2	s_2



Round 1



Round 2

DEFERRED ACCEPTANCE

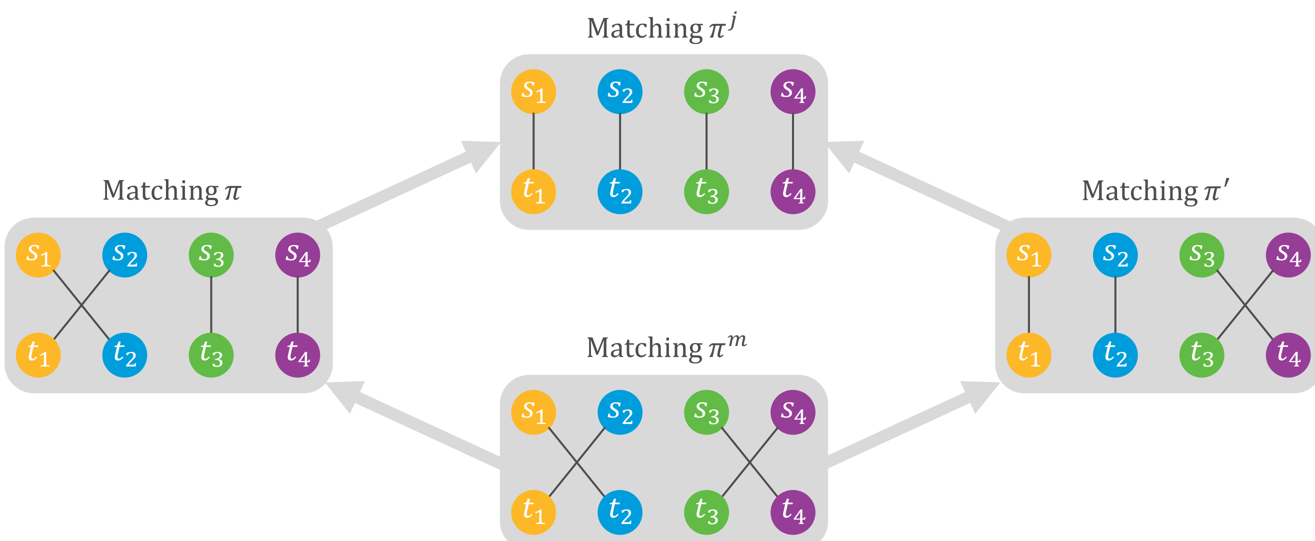
- Theorem: The student-proposing DA algorithm terminates with a stable matching
- Proof:
  - DA terminates because for each round $\ell > 1$, at least one proposal was rejected in the previous round and no student repeats a proposal
  - Suppose that in the final matching π, (s, t) and (s', t') are paired, and (s, t') is a blocking pair
  - Since $t' \succ_{\sigma_s} t$, s proposed to t' before t, implying that $s' \succ_{\sigma_{t'}} s$ — a contradiction ■

THE LATTICE PROPERTY

- Define the student-respecting preference ordering $\pi \geq_S \pi'$ to mean that $\pi(s) \succcurlyeq_{\sigma_s} \pi'(s)$ for all $s \in S$ (with equality iff $\pi = \pi'$)
- Where it exists, define the join $\pi^j = \pi \vee \pi'$ as a stable matching π^j such that $\pi^j \geq_S \pi, \pi^j \geq_S \pi'$, and for every stable $\pi^\star$ satisfying these inequalities, $\pi^\star \geq_S \pi^j$
- Where it exists, define the meet $\pi^m = \pi \wedge \pi'$ as a stable matching π^m such that $\pi^m \leq_S \pi$, $\pi^m \leq_S \pi'$, and for every stable $\pi^\star$ satisfying these inequalities, $\pi^\star \leq_S \pi^m$
- Theorem: The join and meet exist for any pair of stable matchings

THE LATTICE PROPERTY: EXAMPLE

s_1	s_2	s_3	s_4	t_1	t_2	t_3	t_4
t_1	t_2	t_3	t_4	s_4	s_3	s_2	s_1
t_2	t_1	t_4	t_3	s_3	s_4	s_1	s_2
t_3	t_4	t_1	t_2	s_2	s_1	s_4	s_3
t_4	t_3	t_2	t_1	s_1	s_2	s_3	s_4



PROOF OF THEOREM

- Define a pointing operator λ such that, for two matchings π and π', returns as $\lambda(s)$ whichever of $\pi(s)$ and $\pi'(s)$ is more preferred by s, and as $\lambda(t)$ whichever of $\pi(t)$ and $\pi'(t)$ is less preferred by t
- We will prove that given two stable matchings π and π', $\lambda = \pi \vee_S \pi'$
- A symmetric definition of the pointing operator and analogous proof show the existence of the meet

PROOF OF THEOREM

- λ is a matching:
  - Suppose (s, t) are matched in π and $(s, t'), (s', t)$ in π', and w.l.o.g. $\lambda(s) = t$
  - It holds that $\lambda(t) = s$, as otherwise (s, t) is a blocking pair in π'
- λ is stable:
  - Suppose (s, t) is blocking in λ
  - W.l.o.g., suppose $\lambda(t) = \pi(t)$, then $s \succ_{\sigma_t} \pi(t)$ by blocking
  - $t \succ_{\sigma_s} \lambda(s)$ by blocking and $\lambda(s) \succcurlyeq_{\sigma_s} \pi(s)$ by definition
  - Hence, (s, t) is blocking in π — a contradiction
- λ is the join because every matching $\pi^\star$ satisfying $\pi^\star \geq_S \pi$ and $\pi^\star \geq_S \pi'$ must at least take the student-wise max ■

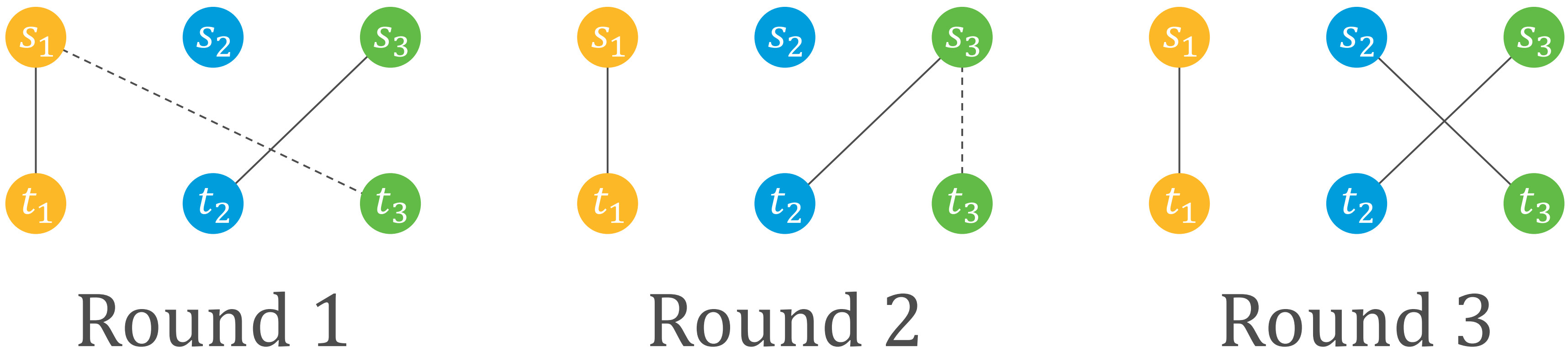
OPTIMAL STABLE MATCHINGS

- By the definition of the pointing operator, moving up in the lattice (join) is better for students and worse for courses, and moving down (meet) is worse for students and better for courses
- It follows that there exist:
  - A student-optimal (and course-pessimal) stable matching $\bar{\pi}$ such that $\bar{\pi} \geq_S \pi$ for every stable matching π
  - A student-pessimal (and course-optimal) stable matching $\underline{\pi}$ such that $\underline{\pi} \leq_S \pi$ for every stable matching π
- Theorem: The student-proposing DA terminates with a student-optimal stable matching and the course-proposing DA terminates with a course-optimal stable matching

COURSE-PROPOSING DA

s_1	s_2	s_3
t_2	t_1	t_1
t_1	t_3	t_2
t_3	t_2	t_3

t_1	t_2	t_3
s_1	s_3	s_1
s_3	s_1	s_3
s_2	s_2	s_2



PROOF OF THEOREM

- Since students propose by order of decreasing preference, if s is matched with t, s was rejected by all t' such that $t' \succ_{\sigma_s} t$, so it suffices to show that no student is ever rejected by an achievable course
- We prove this by induction on the number of rounds, where the base case of $\ell = 1$ is trivial
- In round ℓ, suppose t rejects s in favor of s', so $s' \succ_{\sigma_t} s$
- By the induction assumption, s' prefers t to every achievable course
- If there was a stable matching π with $\pi(s) = t, \pi(s') = t'$ for an achievable t', then (s', t) would be a blocking pair in π — a contradiction ∎

INCENTIVES

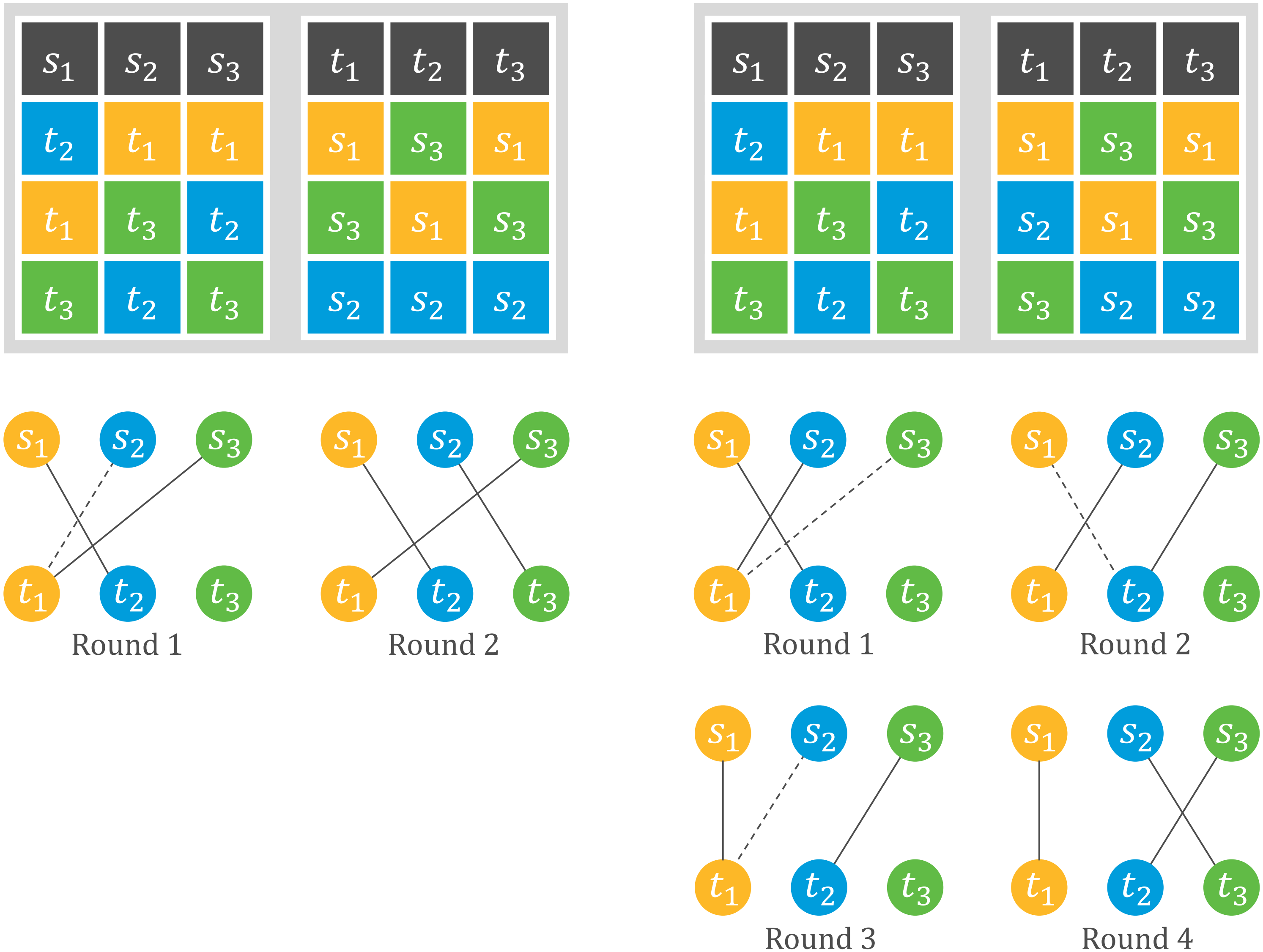
- What happens when students misreport their preferences?

Poll

Student-proposing DA is truthful for:

- ○ Students
- ○ Courses
- ○ Both sides
- ○ Neither side

INCENTIVES



INCENTIVES

- Theorem: Truthful reporting is a dominant strategy for students in student-proposing DA
- Theorem: No bipartite matching mechanism with two-sided preferences is strategyproof (on both sides) and stable
- Assignment 4 asks you to prove the latter theorem

STABLE MATCHING IN PRACTICE

School choice

Match students
with schools

Resident matching

Match residents
with hospitals