ECONOMICS & COMPUTATION

Spring 2025 | Lecture 14

Kidney Exchange

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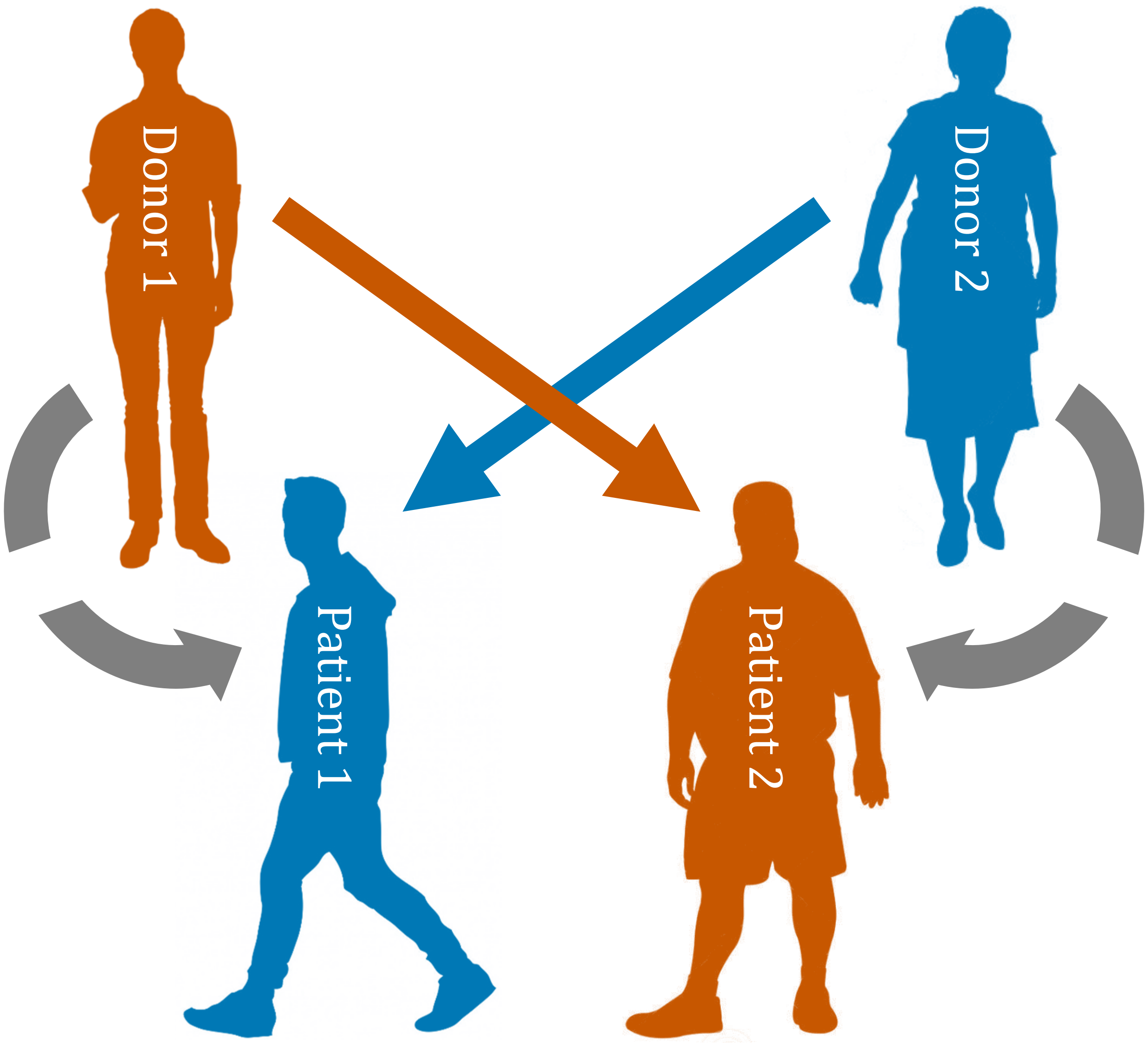
KIDNEY EXCHANGE

- Kidney failure is a serious medical problem
- The preferred treatment is kidney transplant, from a deceased or live donor
- Must be blood-type and tissue-type compatible
- On March 25, 2025, there were 90,489 patients waiting for kidney transplant in the United States
  https://optn.transplant.hrsa.gov/data

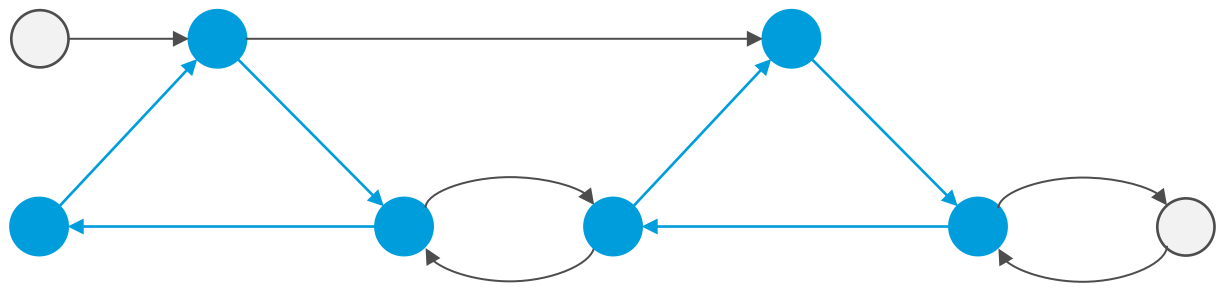
KIDNEY EXCHANGE



KIDNEY EXCHANGE GRAPHS

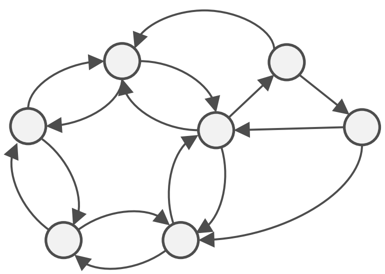
- More generally, we could have exchanges along longer cycles, although length 3 is usually the practical limit
- Model as a directed graph $G = (V, E)$ where V is a set of donor-patient pairs and there is an edge from u to v if the donor of u is compatible with the patient of v

CYCLE COVER

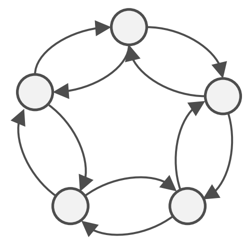


- Cycle-Cover problem: Given a directed graph G and $L \in \mathbb{N}$, find a collection of disjoint cycles of length $\leq L$ in G that maximizes the number of covered vertices
- Theorem: For any constant $L \geq 3$, Cycle-Cover is NP-complete

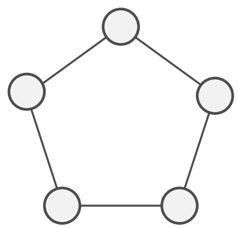
CYCLE-COVER WITH $L = 2$



Given a graph with cycles of any length



Focus on cycles of length 2



Now it's a max matching problem!

KIDNEY EXCHANGE AS IP

- For $L \geq 3$, CYCLE-COVER is solved in practice via integer programming
- Variables: For each cycle c of length $\ell_c \leq L$, variable $x_c \in \{0,1\}$, $x_c = 1$ iff cycle c is included in the cover

$$\begin{aligned} &\max \textstyle\sum_c x_c \ell_c \\ &\text{s.t.} \quad \forall v \in V, \textstyle\sum_{c:v \in c} x_c \leq 1 \\ &\qquad\ \ \forall c, x_c \in \{0,1\} \end{aligned}$$

APPLICATION: UNOS

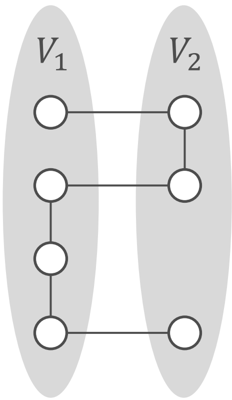
INCENTIVES

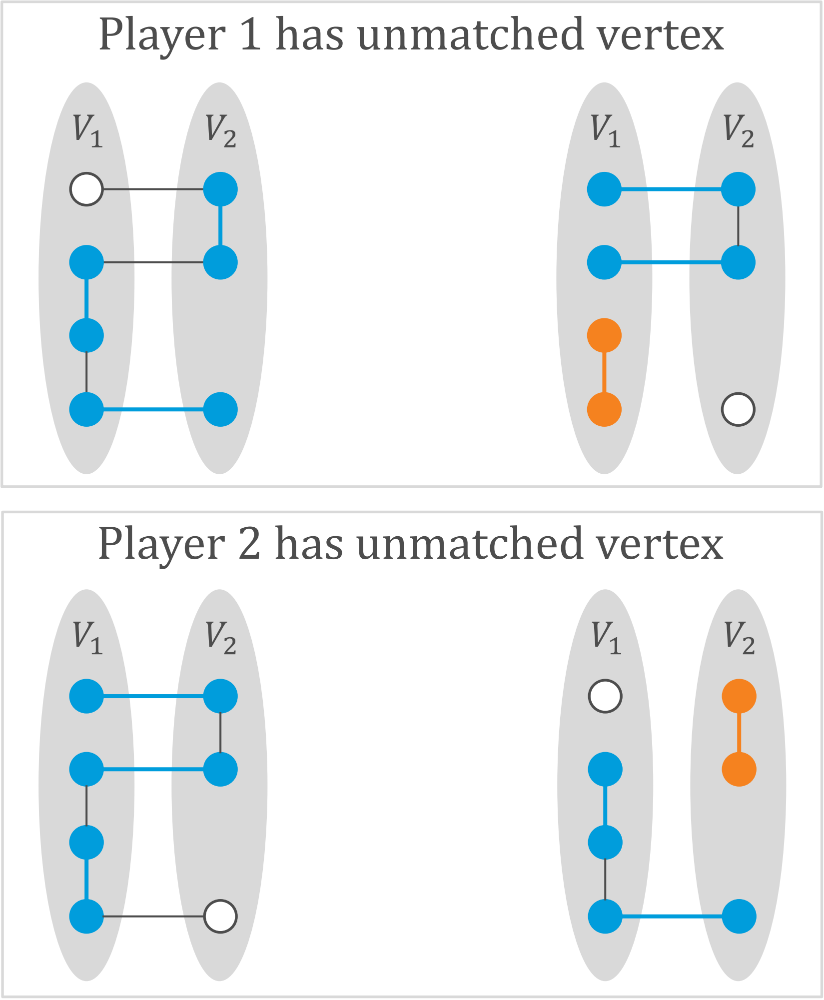
- In the past kidney exchanges were carried out by individual hospitals
- Today there are nationally organized exchanges; participating hospitals have little other interaction
- It was observed that hospitals match easy-to-match pairs internally, and enroll only hard-to-match pairs into larger exchanges
- Goal: incentivize hospitals to enroll all their pairs

THE STRATEGIC MODEL

- Undirected graph (only pairwise matches!)
- Each player controls subset of vertices
- Mechanism receives a graph and returns a matching
- Utility of player = # its matched vertices
- Target: # matched vertices (util. social welfare)
- Strategy: subset of revealed vertices (but edges are public knowledge)
- Mechanism is strategyproof (SP) if it is a dominant strategy to reveal all vertices

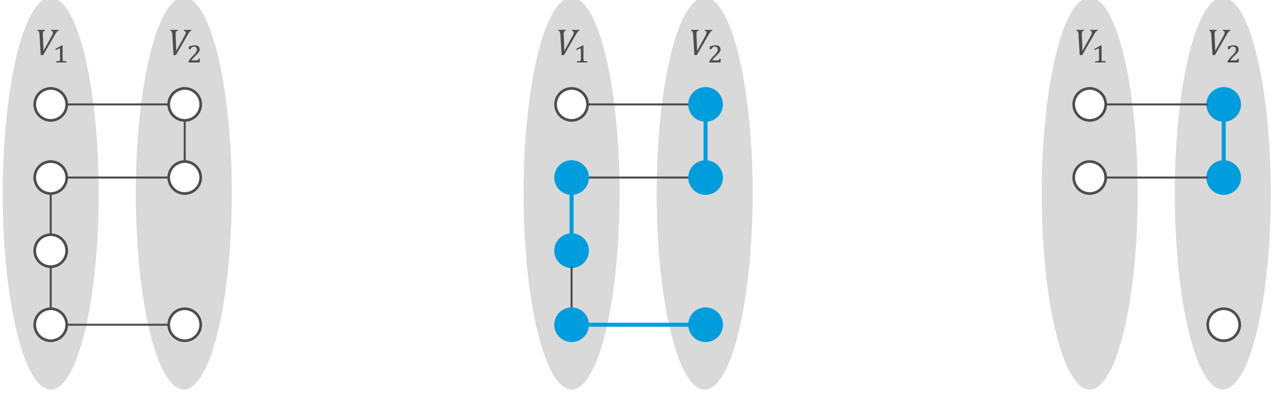
OPT IS MANIPULABLE





SP MECHANISM: TAKE 1

- Assume two players
- The $\textsc{Match}_{\{\{1\},\{2\}\}}$ mechanism:
  - Consider matchings that maximize the number of "internal edges"
  - Among these return a matching with max cardinality

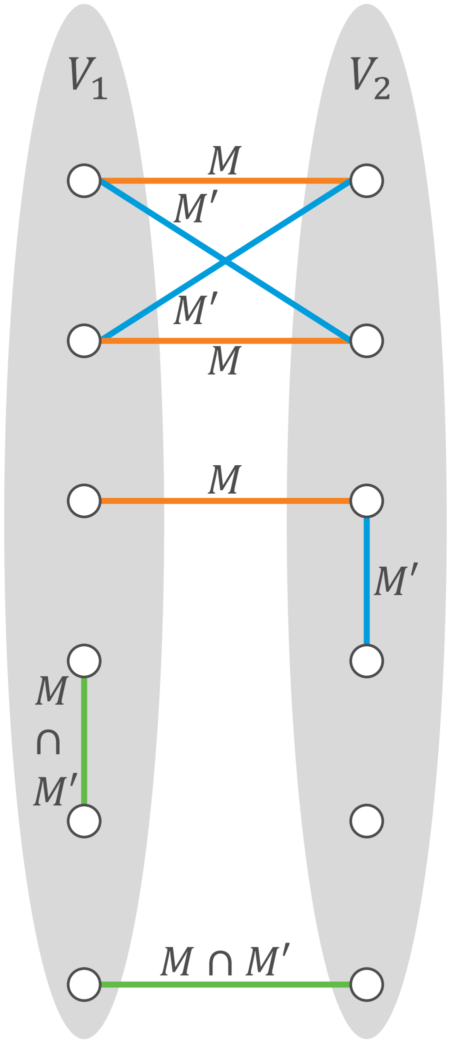


GUARANTEES

- $\textsc{Match}_{\{\{1\},\{2\}\}}$ gives a 1/2-approximation
  - Cannot add more edges to matching
  - For each edge in optimal matching, one of the two vertices is in mechanism's matching
- Theorem (special case): $\textsc{Match}_{\{\{1\},\{2\}\}}$ is strategyproof for two players

PROOF OF THEOREM

- M = matching when player 1 is honest, M' = matching when player 1 hides vertices
- $M \Delta M'$ consists of paths and cycles, each consisting of alternating M, M' edges
- In a cycle, M and M' both match all vertices, so player 1 is indifferent
- We will show that for every path, player 1 has at least as many matched vertices under M as M'



PROOF OF THEOREM

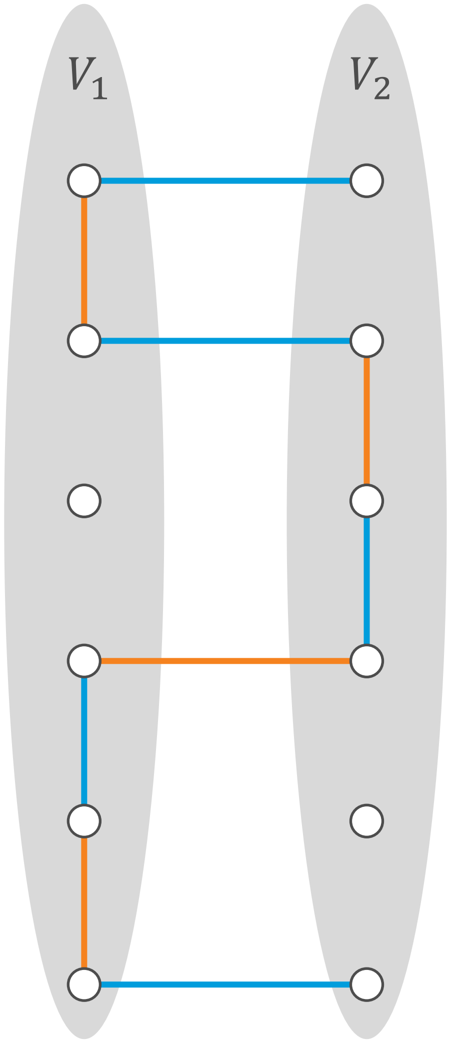
- Consider a path in $M \Delta M'$, denote its edges in M by P and its edges in M' by P'
- For $i, j \in \{1,2\}$,

$$P_{ij} = \left\{(u, v) \in P : u \in V_i, v \in V_j\right\}$$
$$P'_{ij} = \left\{(u, v) \in P' : u \in V_i, v \in V_j\right\}$$

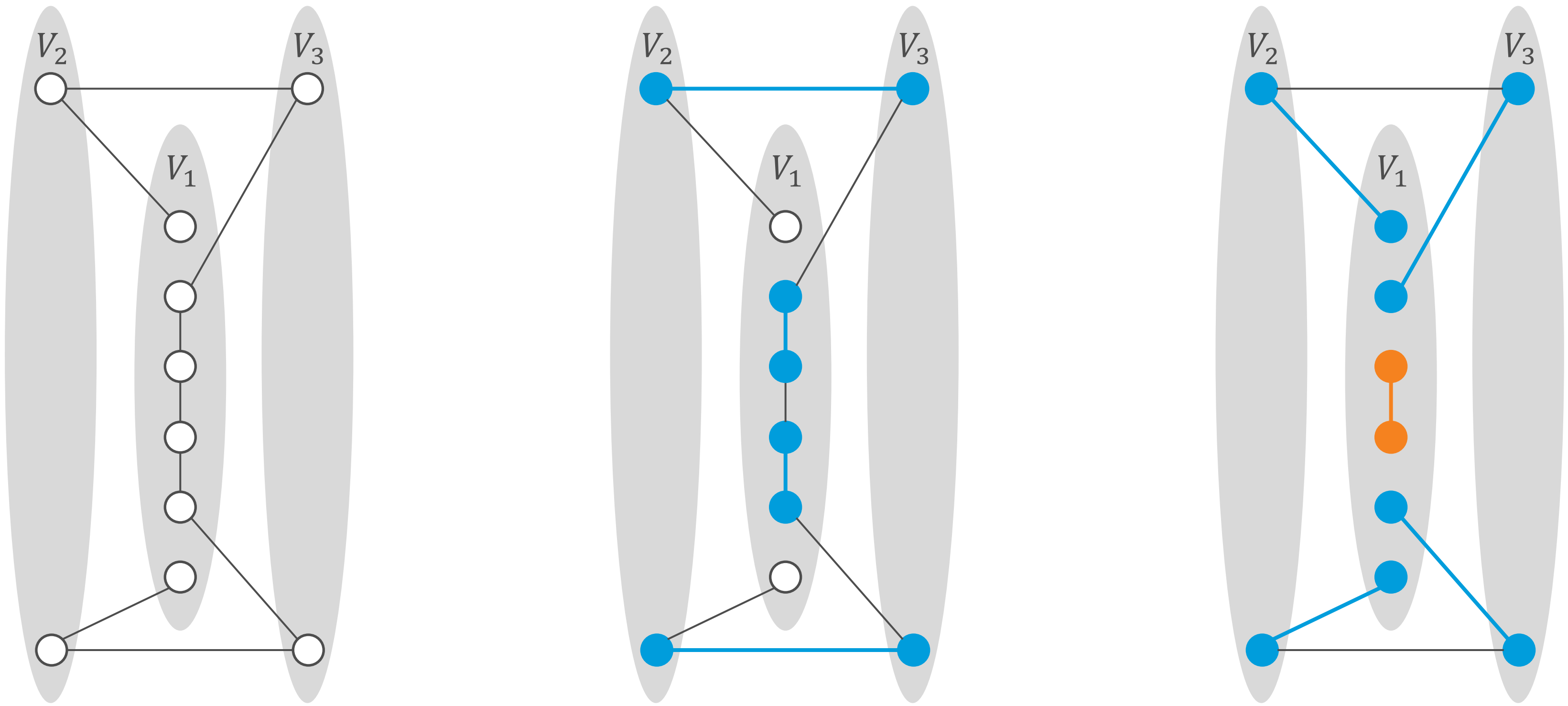
- $|P_{11}| \geq |P'_{11}|$, suppose $|P_{11}| = |P'_{11}|$
- It holds that $|P_{22}| = |P'_{22}|$
- M is max cardinality $\Rightarrow |P_{12}| \geq |P'_{12}|$
- $U_1(P) = 2|P_{11}| + |P_{12}| \geq 2|P'_{11}| + |P'_{12}| = U_1(P')$

PROOF OF THEOREM

- Suppose $|P_{11}| > |P'_{11}|$
- $|P_{12}| \geq |P'_{12}| - 2$
  - Every subpath within V_2 is of even length
  - We can pair the edges of P_{12} and P'_{12}, except maybe the first and the last
- $U_1(P) = 2|P_{11}| + |P_{12}| \geq 2(|P'_{11}| + 1) + |P'_{12}| - 2 = 2|P'_{11}| + |P'_{12}| = U_1(P')$ ■



THE CASE OF 3 PLAYERS



Maximizing internal edges is no longer SP

SP MECHANISM: TAKE 2

- Let $\Pi = (\Pi_1, \Pi_2)$ be a bipartition of the players
- The $\textsc{Match}_\Pi$ mechanism:
  - Consider matchings that maximize the number of "internal edges" and do not have any edges between different players on the same side of the partition
  - Among these return a matching with max cardinality

EUREKA?

- **Theorem:** MATCH_{Π} is strategyproof for any number of players and any partition Π
- Recall: for $n = 2$, $\text{MATCH}_{\{\{1\},\{2\}\}}$ guarantees a 1/2-approximation

Poll

Approximation guarantees given by MATCH_{Π} for $n = 3$ and $\Pi = \{\{1\}, \{2,3\}\}$?

- ○ 1/2-approx
- ○ 1/3-approx
- ○ 1/4-approx
- ○ Less than 1/4

MIX AND MATCH

- The MIX-AND-MATCH mechanism:
  - Mix: Choose a random partition Π
  - Match: Execute MATCH_{Π}
- Theorem: MIX-AND-MATCH is strategyproof and guarantees a 1/2-approximation