



Spring 2025 | Lecture 11

Rent Division

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# PROVABLY FAIR SOLUTIONS.

Spliddit offers quick, free solutions to everyday fair division problems, using methods that provide indisputable fairness guarantees and build on decades of research in economics, mathematics, and computer science.



Share Rent



Split Fare



Assign Credit



Divide Goods



Distribute Tasks



Suggest an App

# ONCE UPON A TIME IN JERUSALEM



Nir Ben Moshe



Naomi Sender

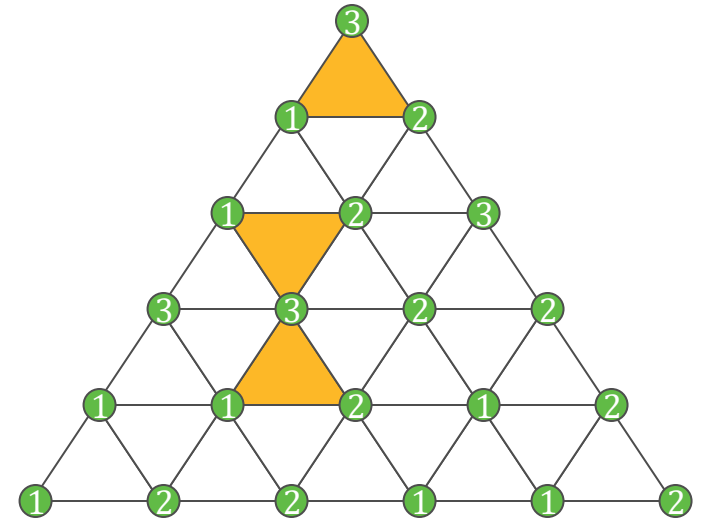


Ariel Procaccia



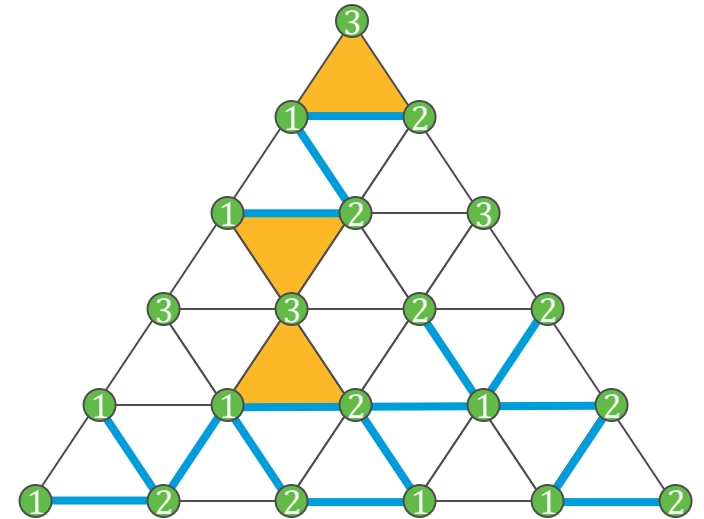
# SPERNER'S LEMMA

- Triangle  $T$  partitioned into **elementary** triangles
- Label vertices by  $\{1,2,3\}$  using **Sperner labeling**:
  - Main vertices are different
  - Label of vertex on an edge  $(i,j)$  of  $T$  is  $i$  or  $j$
- **Lemma:** Any Sperner labeling contains at least one fully labeled elementary triangle



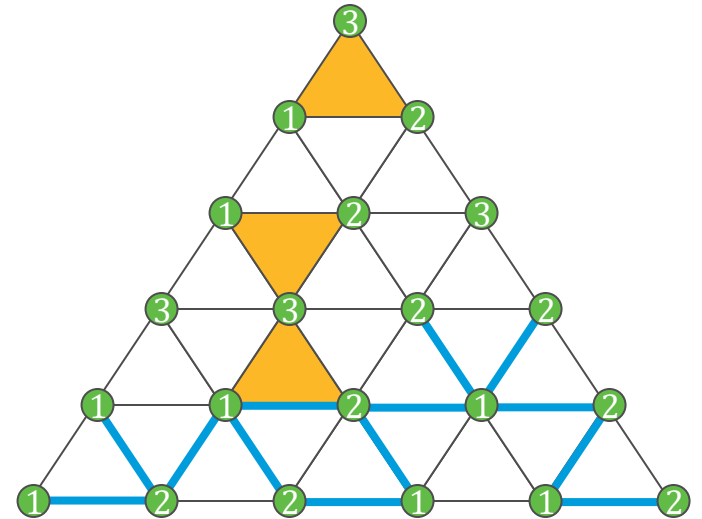
# PROOF OF LEMMA

- Doors are 12 edges
- Rooms are elementary triangles
- #doors on the boundary of  $T$  is odd
- Every room has  $\leq 2$  doors; one door iff the room is 123



# PROOF OF LEMMA

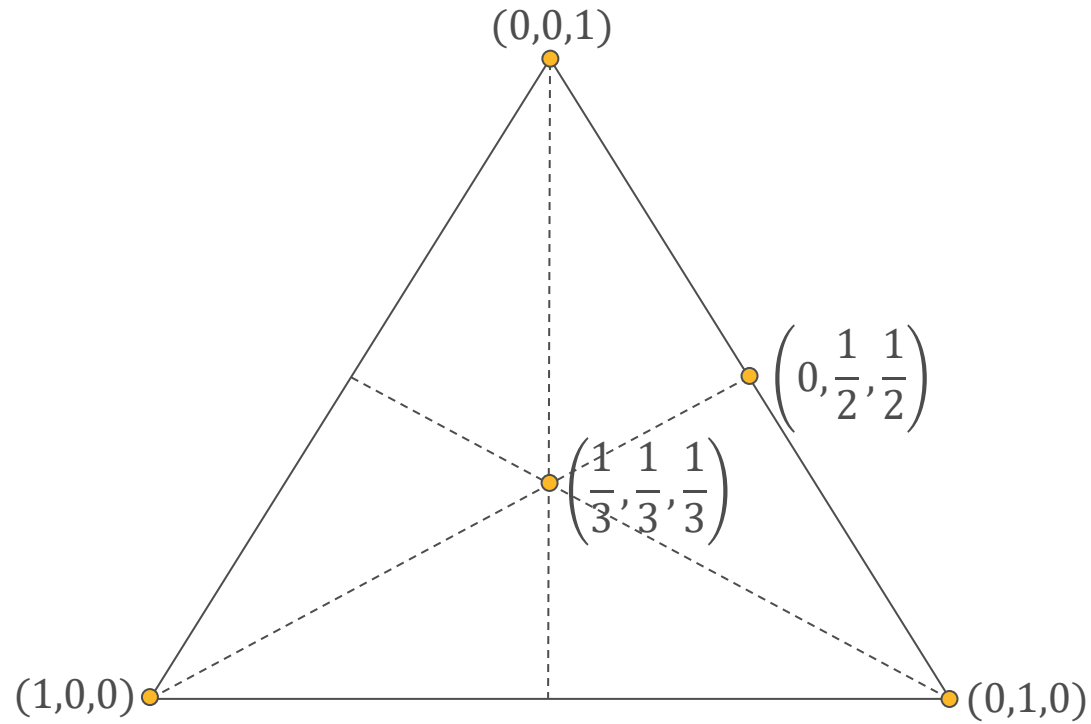
- Start at door on boundary and walk through it
- Room is fully labeled or it has another door...
- No room visited twice
- Eventually walk into fully labeled room or back to boundary
- But #doors on boundary is odd ■



# THE MODEL

- Assume there are three players A, B, C
- Goal is to assign the rooms and divide the rent in a way that is **envy free**: each player prefers their own room at the given prices
- Sum of prices for three rooms is 1
- **Theorem**: An envy-free solution always exists under some assumptions

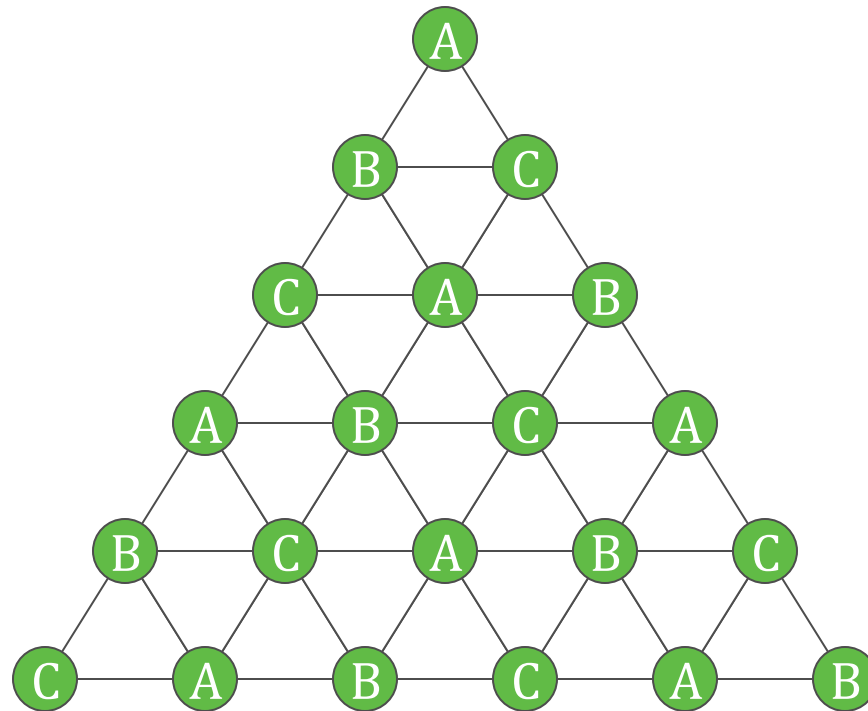
# PROOF OF THEOREM





# PROOF OF THEOREM

- “Triangulate” and assign “ownership” of each vertex to each of A, B, and C, in a way that each elementary triangle is an ABC triangle

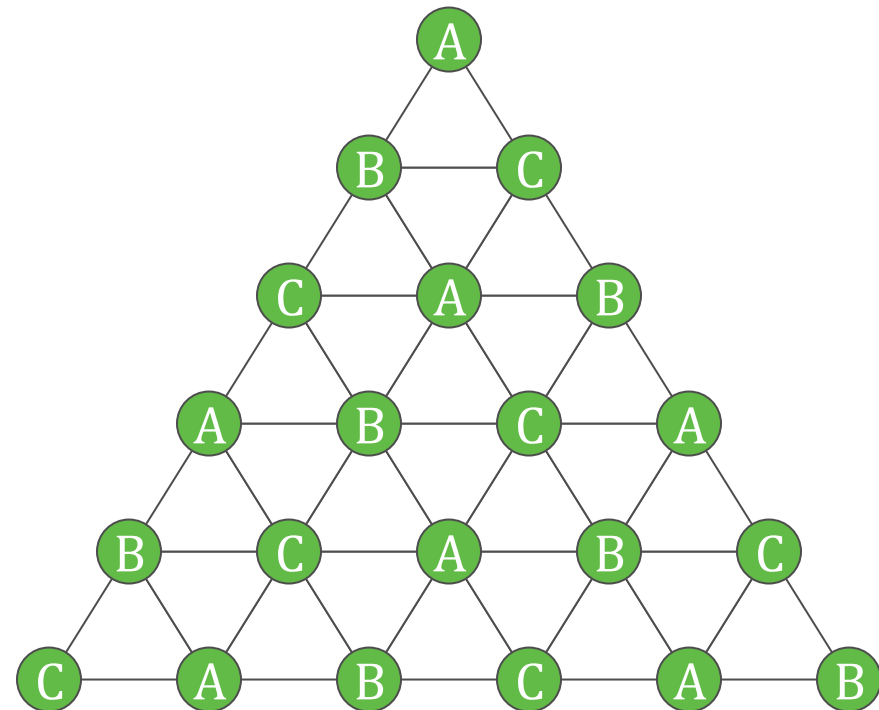
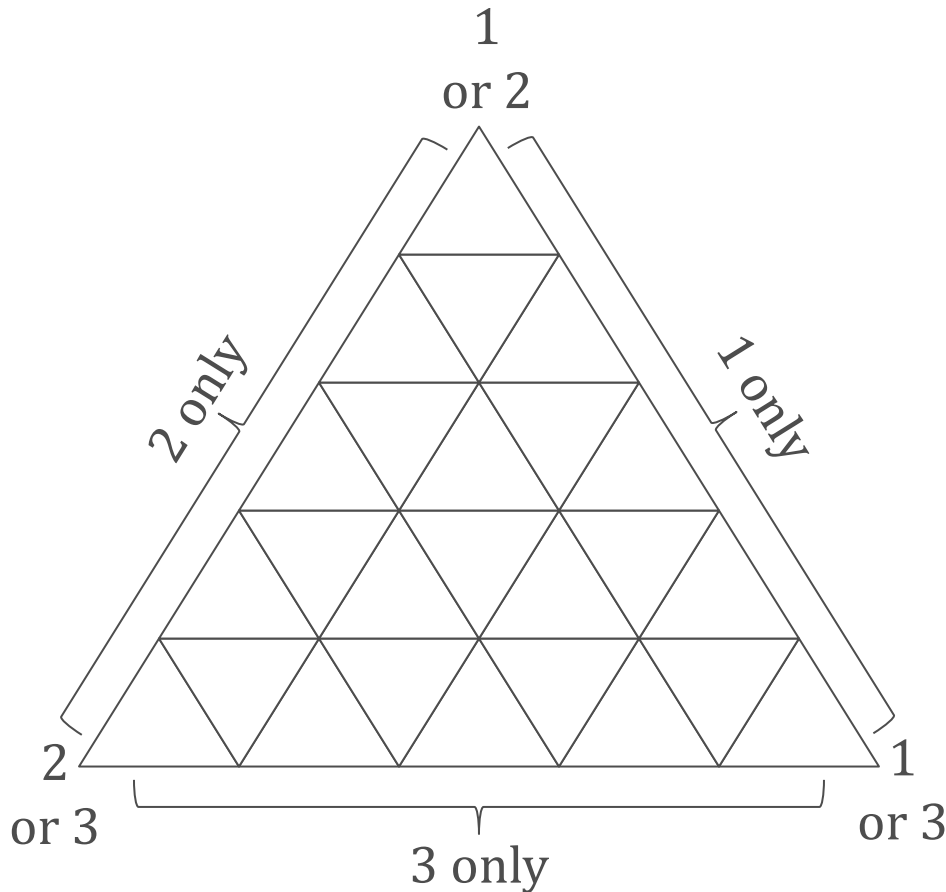


# PROOF OF THEOREM

- Ask the owner of each vertex to tell us which room they prefer
- This gives a new labeling by 1, 2, 3
- Assume that a player wants a free room if one is offered to them

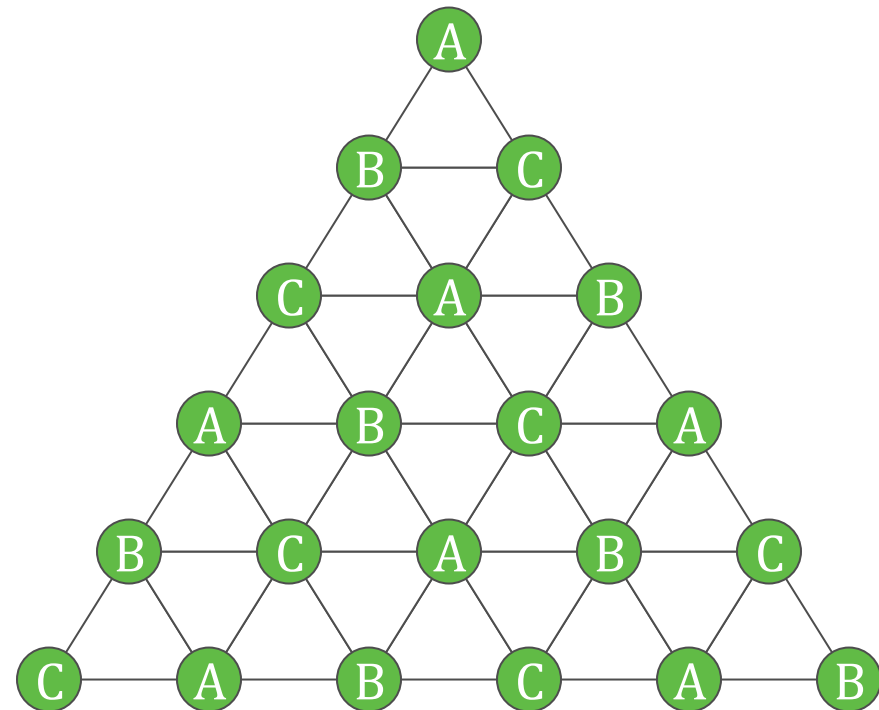
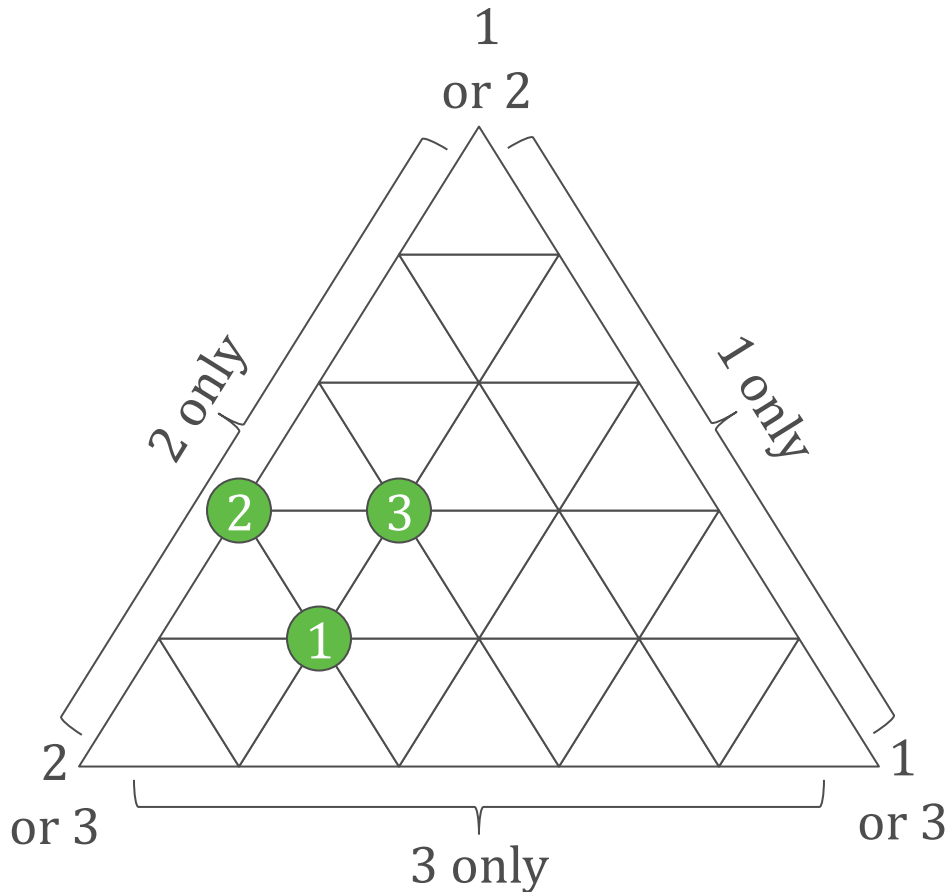
# PROOF OF THEOREM

- Choice of rooms on edges is constrained by free room assumption



# PROOF OF THEOREM

- Sperner's lemma (variant): such a labeling must have a 123 triangle



# PROOF OF THEOREM

- Such a triangle is nothing but an approximately EF solution!
- By making the triangulation finer, we can approach envy-freeness
- Under additional closedness assumption, leads to existence of an EF solution ■

# DISCUSSION

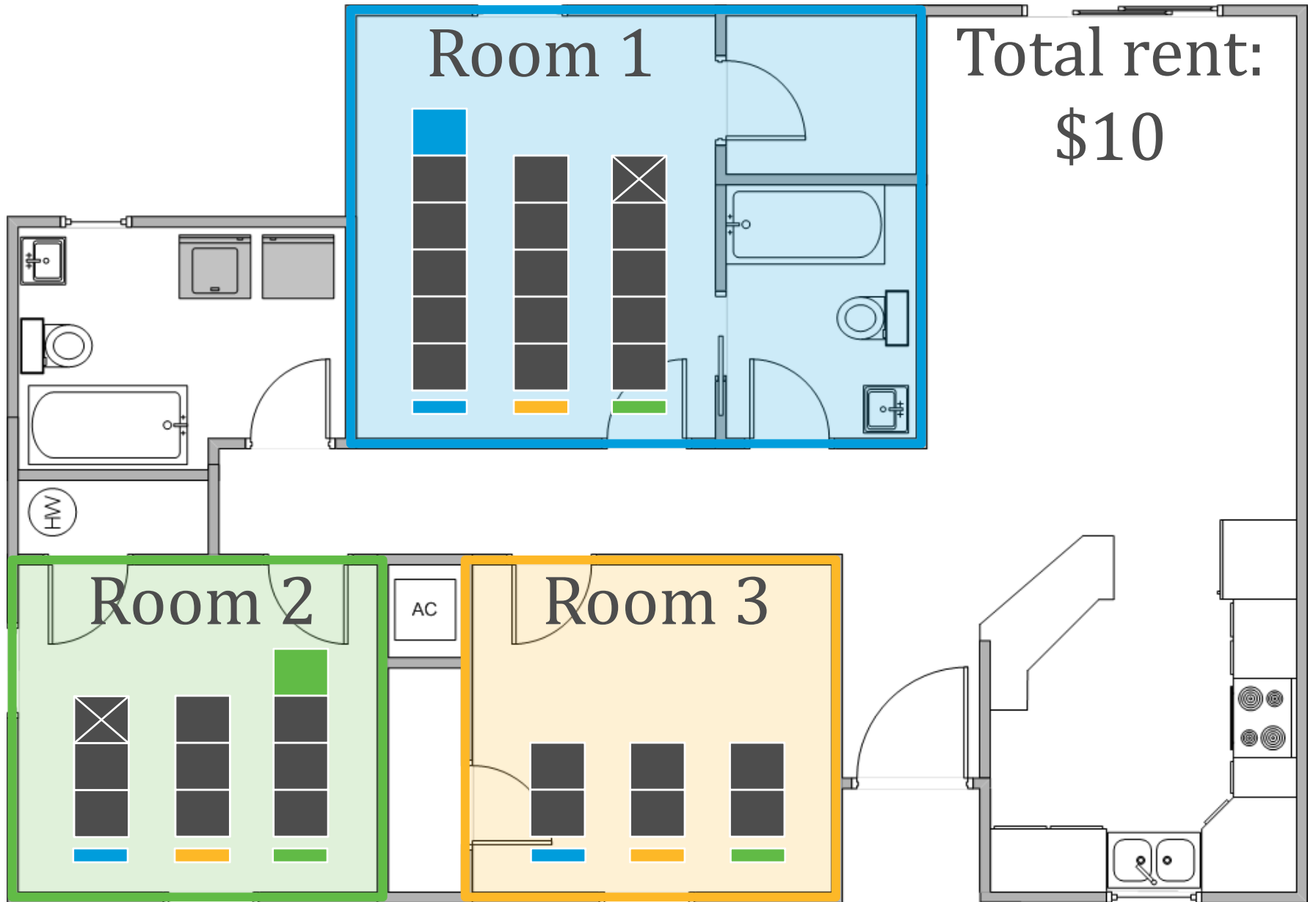
- It is possible to derive an algorithm from the proof
- Same techniques generalize to more players
- Same proof (with the original Sperner's Lemma) shows existence of EF cake division!

# QUASI-LINEAR UTILITIES

- Suppose each player  $i \in N$  has value  $v_{ir}$  for room  $r$
- For all  $i \in N$ ,  $\sum_r v_{ir} = R$ , where  $R$  is the total rent
- The utility of player  $i$  for getting room  $r$  at price  $p_r$  is  $v_{ir} - p_r$
- A **solution** consists of an **assignment**  $\pi$  and a price vector  $\mathbf{p}$ , where  $p_r$  is the price of room  $r$
- Solution  $(\pi, \mathbf{p})$  is **envy free** if and only if
$$\forall i, j \in N, v_{i\pi(i)} - p_{\pi(i)} \geq v_{i\pi(j)} - p_{\pi(j)}$$
- **Theorem:** An envy-free solution always exists under quasi-linear utilities

Room 1

Total rent:  
\$10





# PROPERTIES OF EF SOLUTIONS

- Assignment  $\pi$  is **welfare-maximizing** if

$$\pi \in \operatorname{argmax}_{\sigma} \sum_{i \in N} v_{i\sigma(i)}$$

- **Lemma 1:** If  $(\pi, \mathbf{p})$  is an EF solution, then  $\pi$  is a welfare-maximizing assignment
- **Lemma 2:** If  $(\pi, \mathbf{p})$  is an EF solution and  $\sigma$  is a welfare-maximizing assignment, then  $(\sigma, \mathbf{p})$  is an EF solution

# PROOF OF LEMMA 1

- Let  $(\pi, \mathbf{p})$  be an EF solution, and let  $\sigma$  be another assignment

- Due to EF, for all  $i$ ,

$$v_{i\pi(i)} - p_{\pi(i)} \geq v_{i\sigma(i)} - p_{\sigma(i)}$$

- Summing over all  $i$ ,

$$\sum_{i \in N} v_{i\pi(i)} - \sum_{i \in N} p_{\pi(i)} \geq \sum_{i \in N} v_{i\sigma(i)} - \sum_{i \in N} p_{\sigma(i)}$$

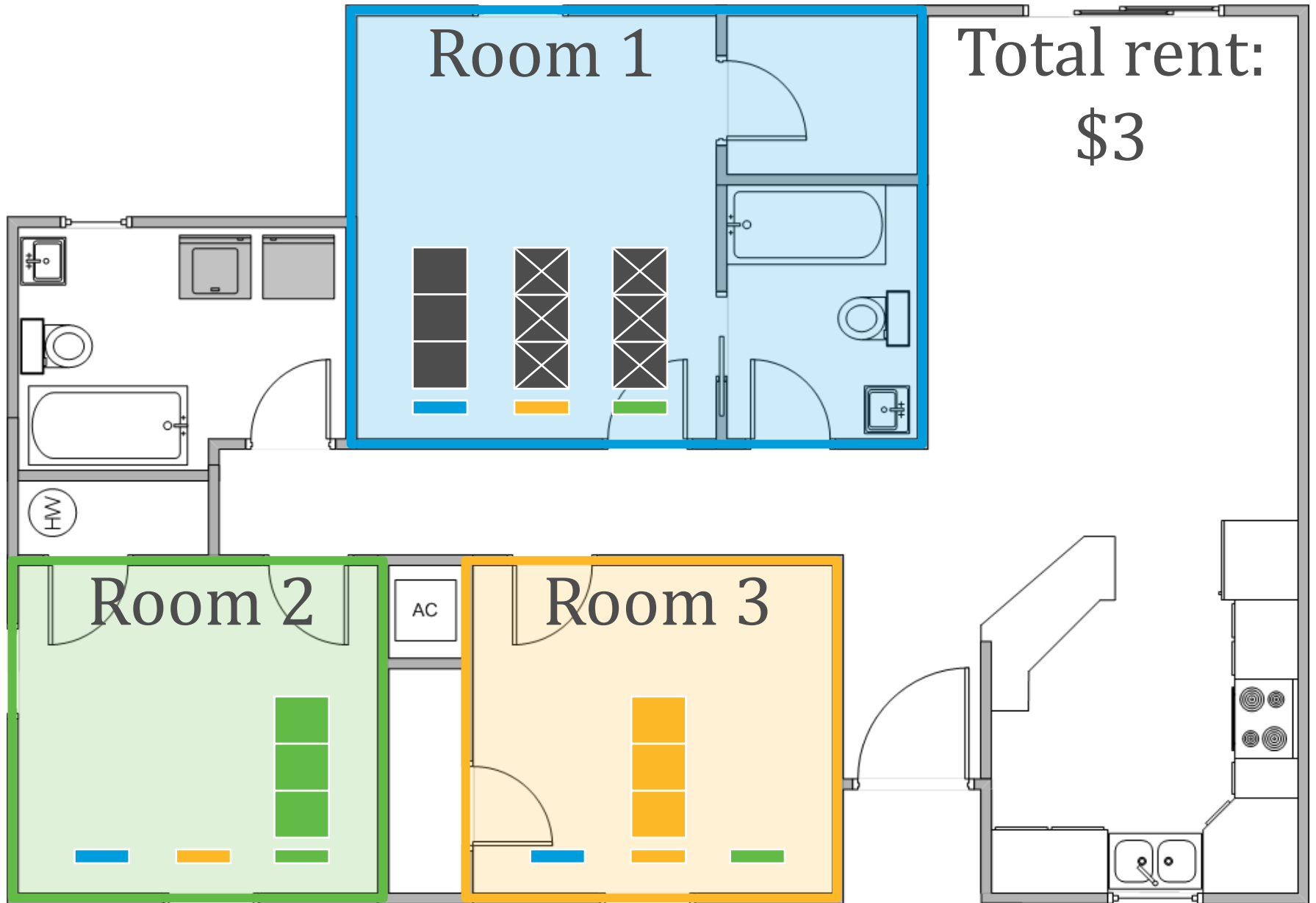
- We get the desired inequality because prices sum up to  $R$  ■

# POLYNOMIAL-TIME ALGORITHM

- Consider the algorithm that finds a welfare-maximizing assignment  $\pi$ , and then finds prices  $\mathbf{p}$  that satisfy the EF constraint
- **Theorem:** The algorithm always returns an EF solution, and can be implemented in polynomial time
- **Proof:**
  - We know that an EF solution  $(\sigma, \mathbf{p})$  exists, by Lemma 2  $(\pi, \mathbf{p})$  is EF, so we would be able to find prices satisfying the EF constraints
  - The first part is max weight matching, the second part is a system of linear inequalities ■

Room 1

Total rent:  
\$3



# OPTIMAL EF SOLUTIONS



## Straw Man Solution

Max sum of utilities  
Subject to envy freeness



## Maximin Solution

Max min utility  
Subject to envy freeness



## Equitable solution

Min max difference in utils  
Subject to envy freeness

# OPTIMAL EF SOLUTIONS

- **Theorem:** The maximin and equitable solutions can be computed in polynomial time
- **Theorem:** The maximin solution is unique
- **Theorem:** The maximin solution is equitable, but not vice versa

# DISCUSSION

- The first model makes no assumptions on utilities other than players preferring free rooms
- The second model assumes quasi-linear utilities

## Poll

Which model do you prefer, the first or the second?



# INTERFACES

## Divide Your Rent Fairly

APRIL 28, 2014

When you're sharing an apartment with roommates, it can be a challenge to decide who takes which bedroom, and at what price. Sit down with your roommates and use the calculator below to find the fair division. [RELATED ARTICLE](#)

What's your total rent? \$ 1000

How many of you are there? 2 3 4 5 6 7 8

If the rooms have the following prices, which room would you choose?

Choices will not necessarily be in order and the same roommate may be asked to choose multiple times in a row. Each roommate keeps choosing until a fair division is found.

Roommate A	\$250 Room 1	\$750 Room 2
Roommate B	<input type="checkbox"/> \$188 Room 1	<input type="checkbox"/> \$813 Room 2

Past Choices

		Room 1	Room 2
All	Roommate B	\$125.00	\$875.00
Roommate A	Roommate B	\$250.00	\$750.00
Roommate B	Roommate B	\$500.00	\$500.00

THE BASICS

+

ALICE'S EVALUATIONS

-

Alice, use the sliders or textboxes to place values on each room. Think of these values as bids: you will never pay more than what you bid, and in most cases you will pay less. However, your values must sum to the total monthly rent: \$1000. You can use the *rescale* button to automatically adjust your values to add up to the rent.

Master Bedroom  .00

Basement  .00

2nd Floor  .00

RESET

RESCALE

CONTINUE

Current Total: \$0

Target: \$1000

BOB'S EVALUATIONS

+

CLAIRE'S EVALUATIONS

+

RESULTS

+

## NY TIMES (rental harmony)

<https://www.nytimes.com/interactive/2014/science/rent-division-calculator.html>

## Spliddit (quasi-linear utilities)

<http://www.spliddit.org/apps/rent>