ECONOMICS & COMPUTATION

Spring 2025 | Lecture 10

Cake Cutting

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CAKE CUTTING

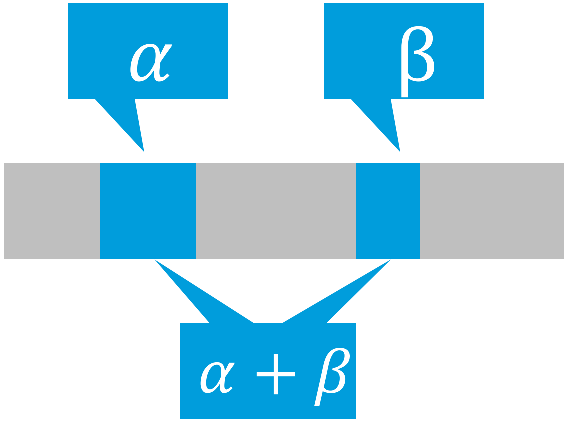
How to fairly divide a heterogeneous divisible good between players with different preferences?

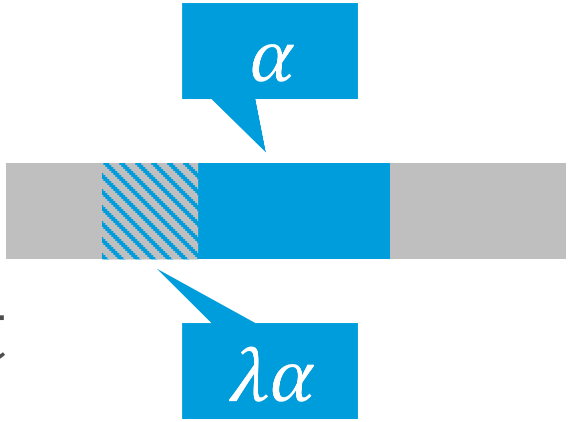
THE PROBLEM

- Cake is interval [0,1]
- Set of players $N = \{1, \dots, n\}$
- Piece of cake $X \subseteq [0,1]$: finite union of subintervals of [0,1]

THE PROBLEM

- Each player $i \in N$ has a non-negative valuation V_i over pieces of cake
- Additive: for $X \cap Y = \emptyset$, $V_i(X) + V_i(Y) = V_i(X \cup Y)$
- Normalized: For all $i \in N$, $V_i([0,1]) = 1$
- Divisible: $\forall \lambda \in [0,1]$ can cut $I' \subseteq I$ s.t. $V_i(I') = \lambda V_i(I)$





FAIRNESS PROPERTIES

- Our goal is to find an allocation $A_1, \ldots, A_n$
- Proportionality:

$$\forall i \in N, V_i(A_i) \geq \frac{1}{n}$$

- Envy-Freeness (EF):

$$\forall i, j \in N, V_i(A_i) \geq V_i(A_j)$$

Poll 1

For $n = 2$, which is stronger?

- Proportionality
- Envy-Freeness
- Equivalent
- Incomparable

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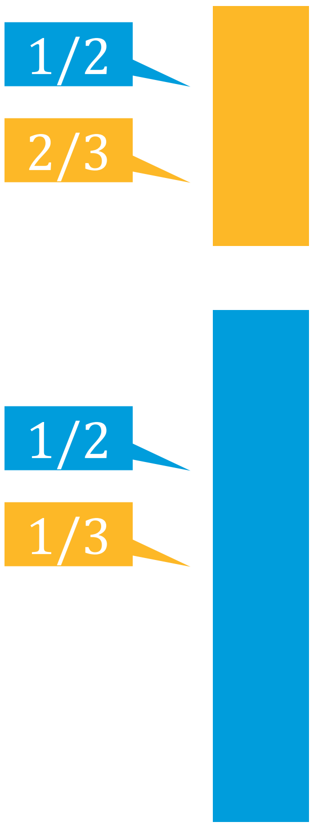
Poll 2

For $n \geq 3$, which is stronger?

- Proportionality
- Envy-Freeness
- Equivalent
- Incomparable

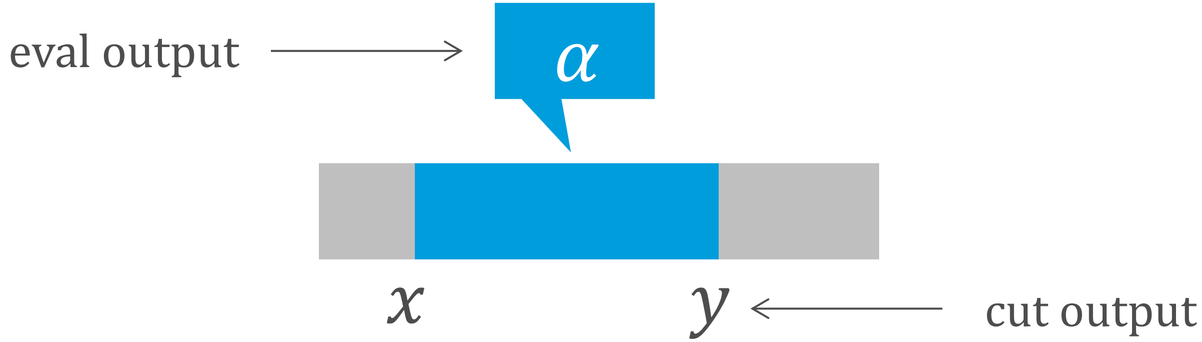
CUT-AND-CHOOSE

- Algorithm for $n = 2$ [Procaccia and Procaccia, circa 1985]
- Player 1 divides into two pieces X, Y s.t. $V_1(X) = 1/2, V_1(Y) = 1/2$
- Player 2 chooses preferred piece
- This is EF (hence proportional)



THE ROBERTSON-WEBB MODEL

- What is the complexity of Cut-and-Choose?
- Input size is n
- Two types of operations
  - $\text{Eval}_i(x, y)$ returns $V_i([x, y])$
  - $\text{Cut}_i(x, \alpha)$ returns y such that $V_i([x, y]) = \alpha$



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Poll 3

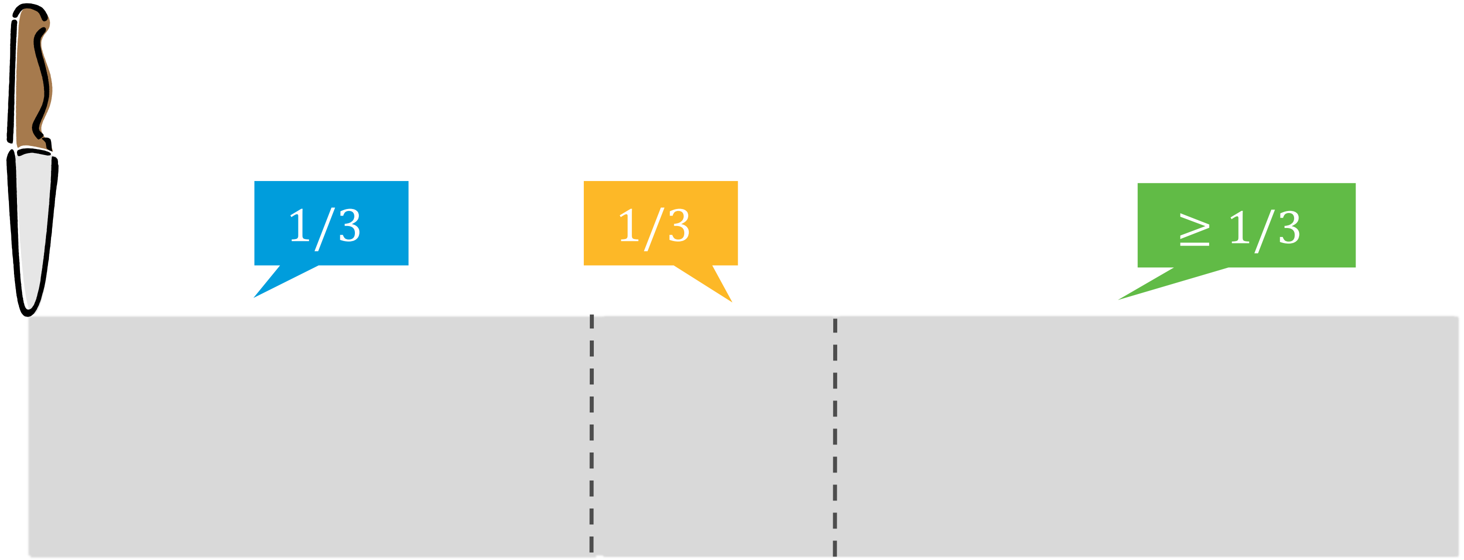
#Operations needed to find an EF allocation when $n = 2$?

- One
- Two
- Three
- Four

DUBINS-SPANIER

- Referee continuously moves knife
- Repeat: when piece left of knife is worth $1/n$ to player, player shouts “stop” and gets piece
- That player is removed
- Last player gets remaining piece

DUBINS-SPANIER PROTOCOL



DUBINS-SPANIER

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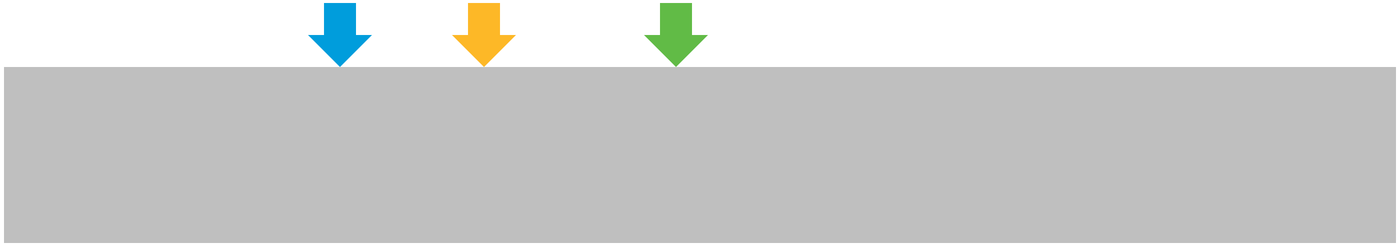
Poll 4

What is the complexity of DS?

- $\Theta(n)$
- $\Theta(n \log n)$
- $\Theta(n^2)$
- $\Theta(n^2 \log n)$

DUBINS-SPANIER



DUBINS-SPANIER

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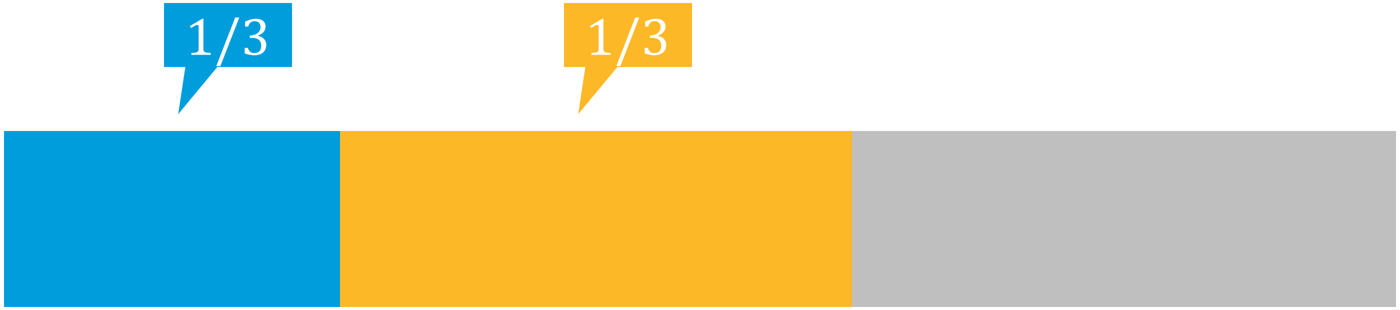
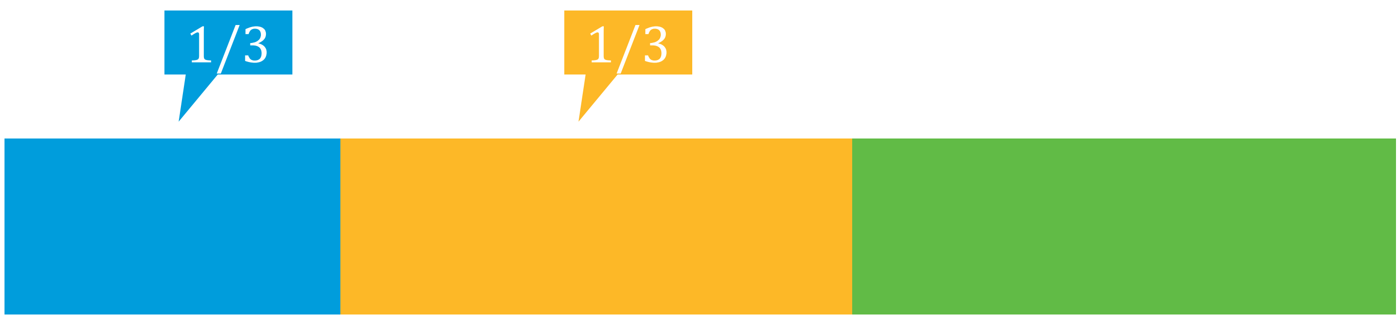


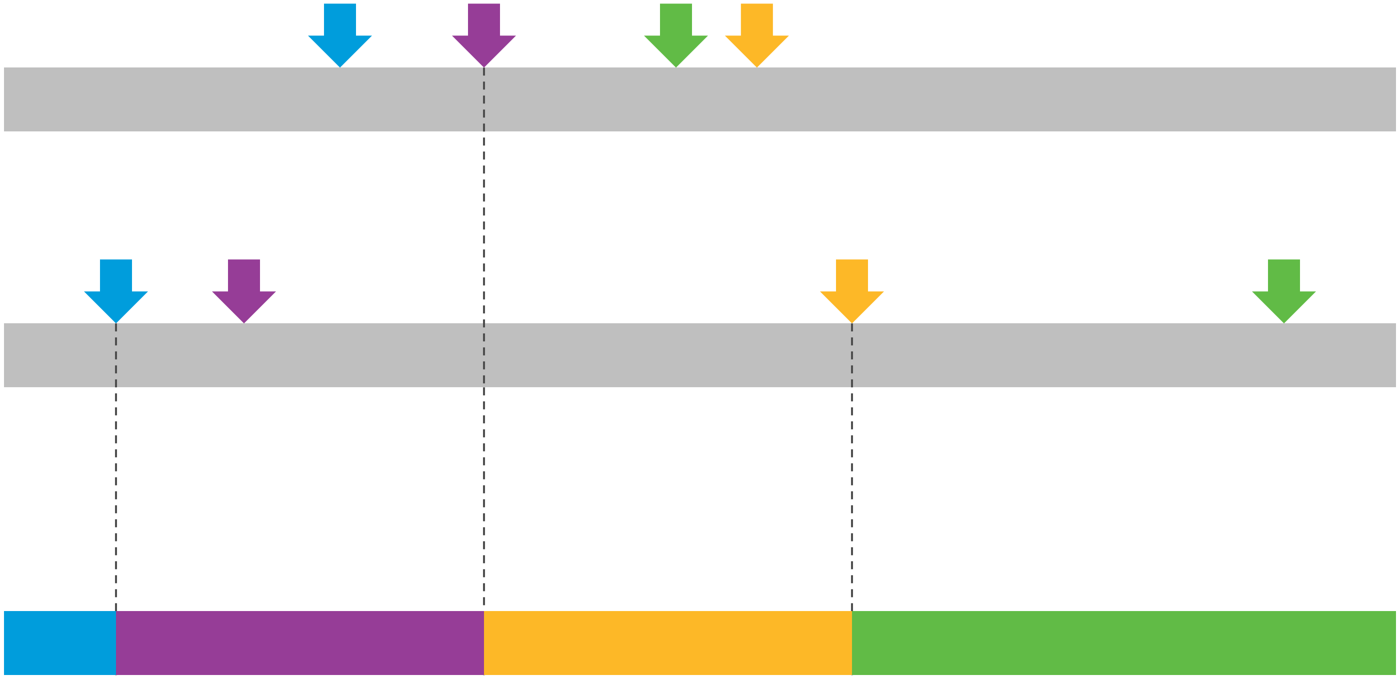
DUBINS-SPANIER



EVEN-PAZ

- Given $[x, y]$, assume $n = 2^k$ for ease of exposition
- If $n = 1$, give $[x, y]$ to the single player
- Otherwise, each player i makes a mark z_i s.t.
$$V_i([x, z_i]) = \frac{1}{2} V_i([x, y])$$
- Let z^* be the $n/2$ mark from the left
- Recurse on $[x, z^*]$ with the left $n/2$ players, and on $[z^*, y]$ with the right $n/2$ players

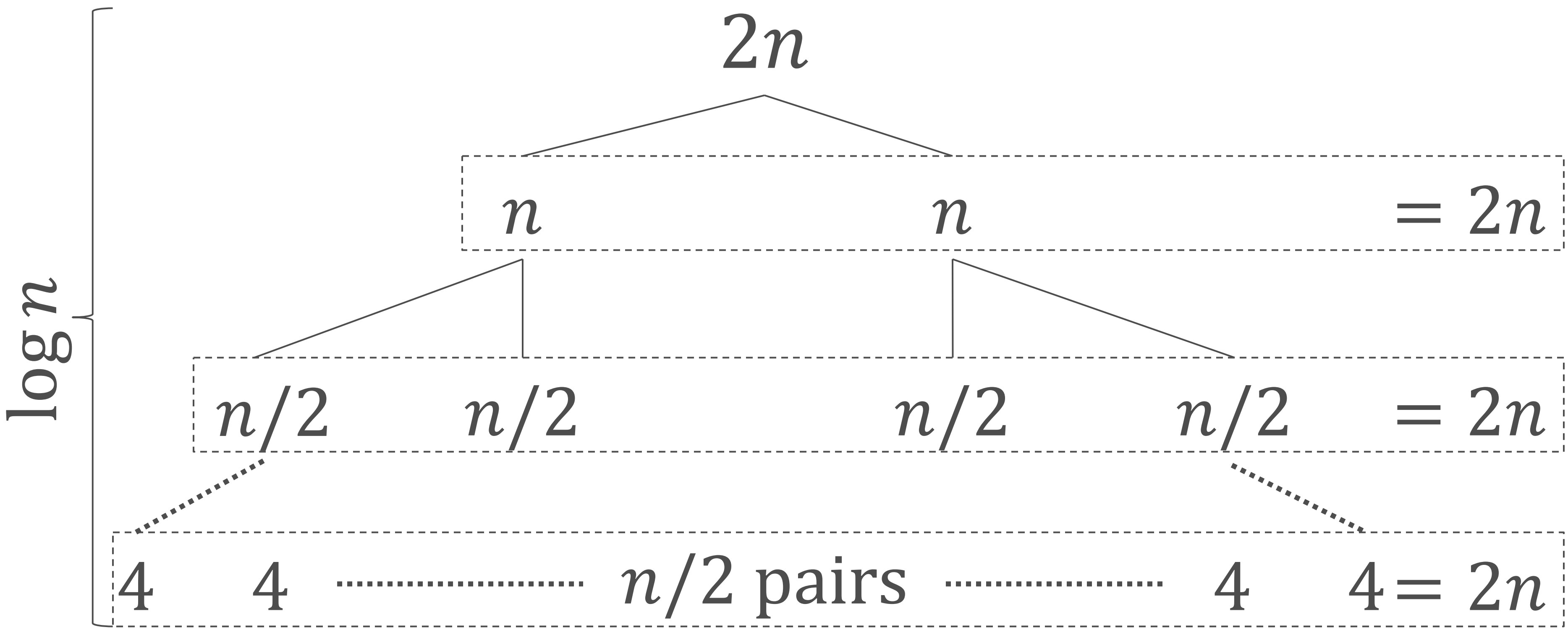
EVEN-PAZ



EVEN-PAZ

- **Theorem:** The Even-Paz protocol produces a proportional allocation
- **Proof:**
  - At stage 0, each of the n players values the whole cake at 1
  - At each stage the players who share a piece of cake value it at least at $V_i([x, y])/2$
  - Hence, if at stage k each player has value at least $1/2^k$ for the piece they're sharing, then at stage $k + 1$ each player has value at least $\frac{1}{2^{k+1}}$
  - The number of stages is $\log n$ ■

$$T(1) = 0, T(n) = 2n + 2T\left(\frac{n}{2}\right)$$



Overall: $2n \log n$

COMPLEXITY OF PROPORTIONALITY

- Theorem: Any proportional protocol needs $\Omega(n \log n)$ operations in the RW model
- The Even-Paz protocol is provably optimal!
- What about envy?

SELFRIDGE-CONWAY

- Stage 0
  - Player 1 divides the cake into three equal pieces according to V_1
  - Player 2 trims the largest piece s.t. there is a tie between the two largest pieces according to V_2
  - Cake 1 = cake w/o trimmings, Cake 2 = trimmings
- Stage 1 (division of Cake 1)
  - Player 3 chooses one of the three pieces of Cake 1
  - If player 3 did not choose the trimmed piece, player 2 is allocated the trimmed piece
  - Otherwise, player 2 chooses one of the two remaining pieces
  - Player 1 gets the remaining piece
  - Denote the player $i \in \{2, 3\}$ that received the trimmed piece by T, and the other by T'
- Stage 2 (division of Cake 2)
  - T' divides Cake 2 into three equal pieces according to $V_{T'}$
  - Players T, 1, and T' choose the pieces of Cake 2, in that order

THE COMPLEXITY OF EF

- Theorem [Brams and Taylor 1995]: There is an EF cake cutting algorithm in the RW model
- But it is unbounded
- Theorem [Aziz and Mackenzie 2016]: There is a bounded EF algorithm for any n, whose complexity is

$$O\left(n^{n^{n^{n^{n^{n}}}}}\right)$$

- Theorem [Procaccia 2009]: Any EF algorithm requires $\Omega(n^2)$ queries in the RW model