ECONOMICS & COMPUTATION

Spring 2025 | Lecture 1

Nash Equilibrium

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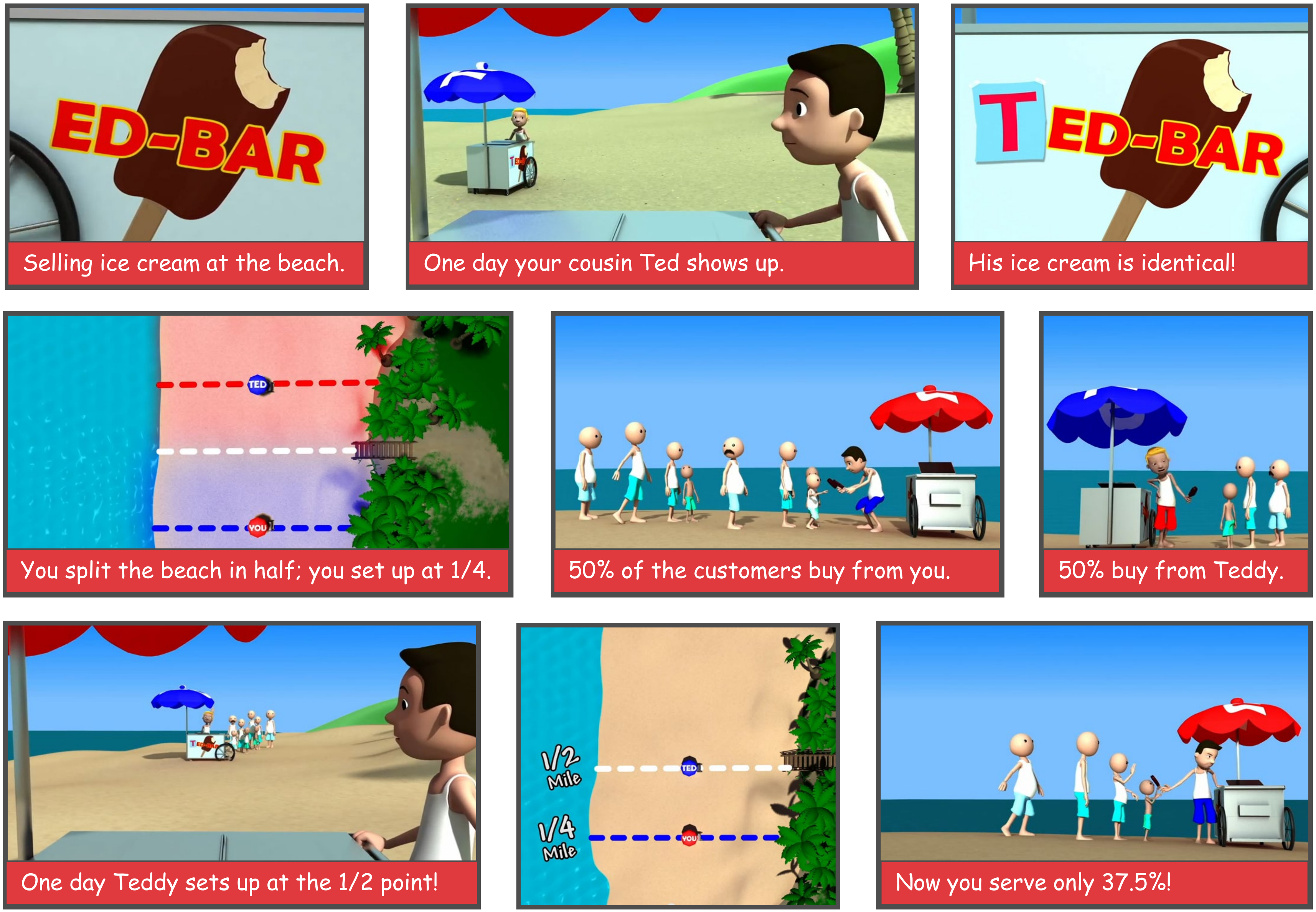
NORMAL-FORM GAME

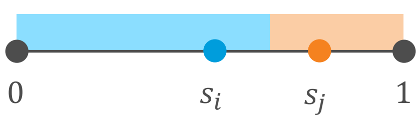
- A game in normal form consists of:
  - Set of players $N = \{1, \ldots, n\}$
  - Strategy set S
  - For each $i \in N$, utility function $u_i: S^n \to \mathbb{R}$, which gives the utility of player i, $u_i(s_1, \ldots, s_n)$, when each $j \in N$ plays the strategy $s_j \in S$
- Next example created by taking screenshots of `http://youtu.be/jILgxeNBK_8`

THE ICE CREAM WARS

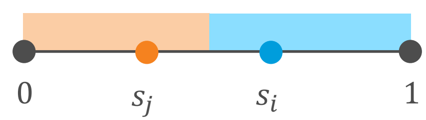


THE ICE CREAM WARS

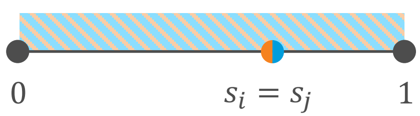
$N = \{1,2\}, S = [0,1],$ and $u_i(s_i, s_j)$ is defined as follows:



$\frac{s_i+s_j}{2}$ if $s_i < s_j$



$1 - \frac{s_i+s_j}{2}$ if $s_i > s_j$



0.5 if $s_i = s_j$

THE PRISONER'S DILEMMA

	Cooperate	Defect
Cooperate	-1,-1	-9,0
Defect	0,-9	-6,-6

What would you do?

UNDERSTANDING THE DILEMMA

- Defection is a dominant strategy
- But the players can do much better by cooperating
- Related to the tragedy of the commons

THE TRAGEDY OF THE COMMONS

Opinion
Ariel Procaccia

Tech Giants, Gorging on AI Professors Is Bad for You

If industry keeps hiring the cutting-edge scholars, who will train the next generation of innovators in artificial intelligence?

Eat too much and there won't be grass for anyone. *Photographer: William West/AFP/Getty Images*

By Ariel Procaccia
January 7, 2019 at 6:00 AM EST

THE PRISONER'S DILEMMA ON TV

http://youtu.be/S0qjK3TWZE8

THE PROFESSOR'S DILEMMA

Professor \ Class	Listen	Sleep
Make effort	10^6,10^6	-10,0
Slack off	0,-10	0,0

Dominant strategies?

John Forbes Nash

1928–2015

Mathematician and Nobel laureate in economics. Also remembered as the protagonist in “A Beautiful Mind.”

NASH EQUILIBRIUM

- In a Nash equilibrium, no player wants to unilaterally deviate
- Each player's strategy is a best response to strategies of others
- Formally, a Nash equilibrium is a vector of strategies $\boldsymbol{s} = (s_1 \dots, s_n) \in S^n$ such that for all $i \in N, s_i' \in S$,

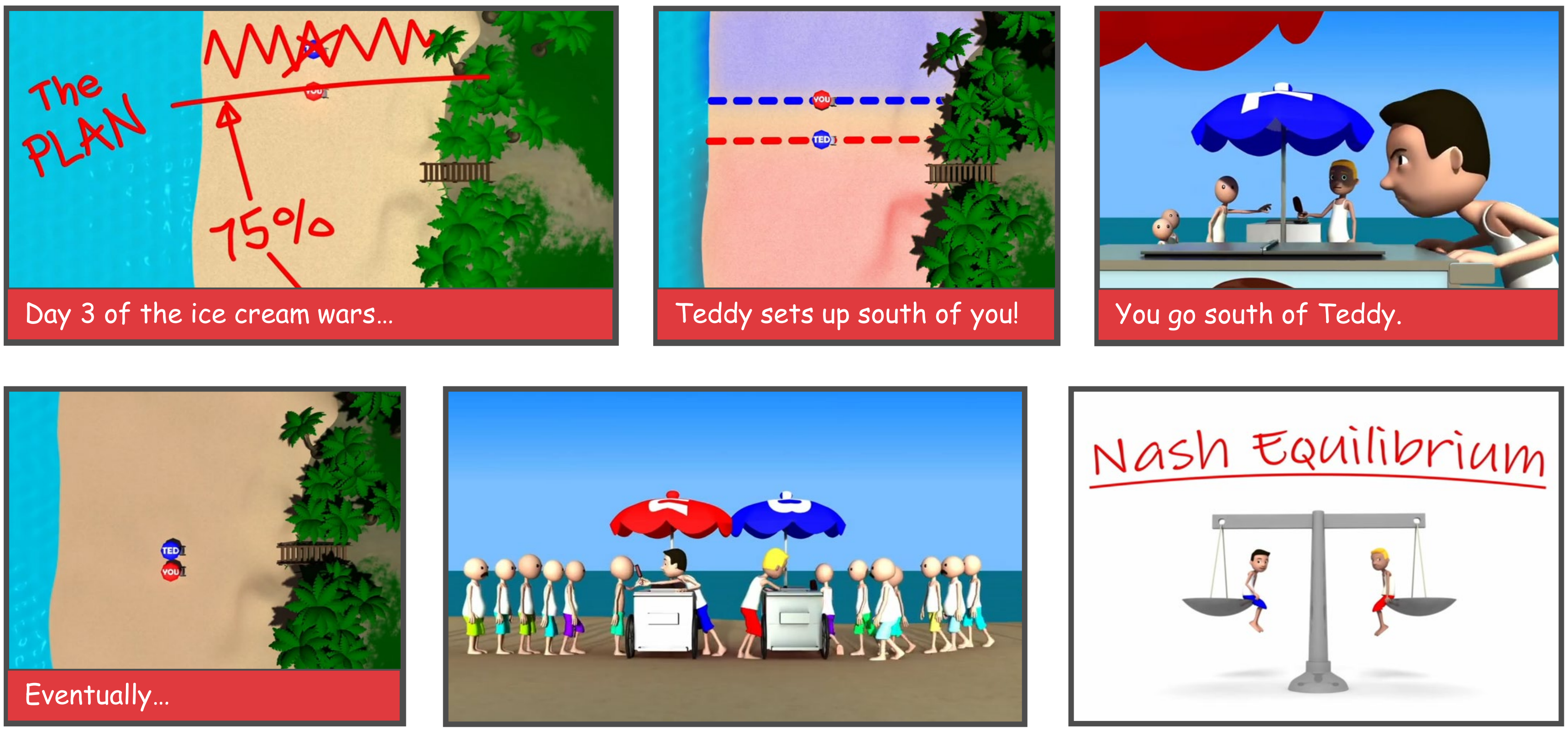
$$u_i(\boldsymbol{s}) \geq u_i(s_i', \boldsymbol{s}_{-i})$$

THE PROFESSOR'S DILEMMA

		Class	
		Listen	Sleep
Professor	Make effort	$10^6,10^6$	-10,0
	Slack off	0,-10	0,0

Nash equilibria?

END OF THE ICE CREAM WARS



ROCK-PAPER-SCISSORS

			
	0,0	-1,1	1,-1
	1,-1	0,0	-1,1
	-1,1	1,-1	0,0

Nash equilibria?

MIXED STRATEGIES

- A mixed strategy is a probability distribution over (pure) strategies
- The mixed strategy of player $i \in N$ is x_i, where

$$x_i(s_i) = \Pr[i \text{ plays } s_i]$$

- The utility of player $i \in N$ is

$$u_i(x_1, \dots, x_n) = \sum_{(s_1,\dots,s_n)\in S^n} u_i(s_1, \dots, s_n) \cdot \prod_{j=1}^{n} x_j(s_j)$$

EXERCISE: MIXED NE

- **Exercise:** player 1 plays $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, player 2 plays $\left(0, \frac{1}{2}, \frac{1}{2}\right)$. What is u_1?
- **Exercise:** Both players play $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. What is u_1?

			
	0,0	-1,1	1,-1
	1,-1	0,0	-1,1
	-1,1	1,-1	0,0

EXERCISE: MIXED NE

Poll 1

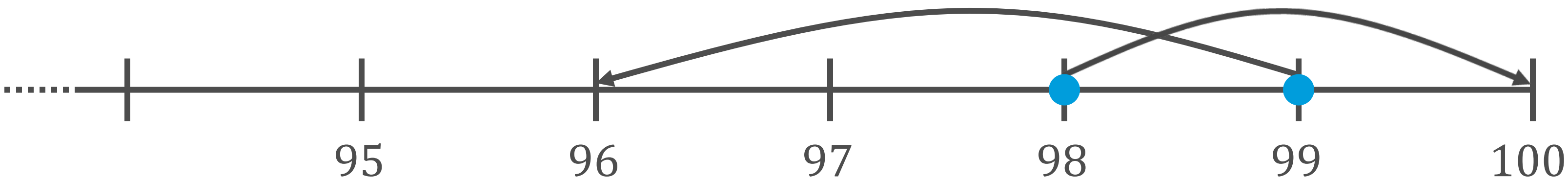
Which is a NE?

- $\left(\left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},\frac{1}{2},0\right)\right)$
- $\left(\left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},0,\frac{1}{2}\right)\right)$
- $\left(\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)$
- $\left(\left(\frac{1}{3},\frac{2}{3},0\right),\left(\frac{2}{3},0,\frac{1}{3}\right)\right)$

	0,0	-1,1	1,-1
	1,-1	0,0	-1,1
	-1,1	1,-1	0,0

Theorem [Nash, 1950]: In any (finite) game there exists at least one (possibly mixed) Nash equilibrium

CAVEAT: NE PREDICTS OUTCOMES?



Two players, strategies are $\{2, \ldots, 100\}$. If both choose the same number, that is what they get. If one chooses s, the other t, and $s < t$, the former player gets $s + 2$, and the latter gets $s - 2$.

Poll 2

Suppose you are paired with another random student, and you must play this game with them (for real money) without communicating. What would you choose?