Harvard CS 1360, Spring 2025
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Matching 4: Random Assignment

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Lecture 16

Assignment Problems: Each player requires exactly one (indivisible) good. EF1 is trivial, since it is always the case that if we remove one good, another player isn't envious.

- **School choice:** Assign students to schools in a supplementary round where only students have preferences.
- **Housing allocation:** Assign applicants to public housing

Definition 1 (Random Assignment Problem)**.** A random assignment model is defined as the following:

- A set of players $N = \{1, \dots, n\}$
- Set G of n goods, assume $|N| = |G|$, i.e., the number of goods and the number of players is identical (for simplicity).
- Each player has a ranking $\sigma_i \in \mathcal{L}$ over the goods G
- An assignment is a perfect matching π between players and goods, where $\pi(i)$ is the good assigned to player i
- We are interested in rules f that take the collection of rankings $\boldsymbol{\sigma} \in \mathcal{L}^n$ and output assignment π.

Definition 2 (Serial Dictatorship)**.** A serial dictatorship mechanism (or sometimes priority mechanism) specifies an order over agents, and then lets the first agent receive their favorite good, the next agent receive their favorite good among remaining objects, etc.

Theorem 1. *Serial dictatorship is strategyproof.*

Proof. Each agent receives their favorite good among the remaining goods, and their preferences do not impact the set of goods available to them. □

Theorem 2. *Serial dictatorship is Pareto efficient.*

Proof. Let π be the output of the algorithm. Suppose for contradiction that the theorem is false, and there is some π' that Pareto-dominates π. Let i be the first agent in the ordering who gets a strictly better good in π' than in π. Then under π, $\pi'(i)$ was not available when when i chose a good. This means that i took the good chosen by some agent j whose turn to pick comes before i. But by assumption, i was the first agent to get a strictly better good, so j has to end up with a worse good. This contradicts that every agent is at least as happy in π' as in π, so π' cannot Pareto-dominate π. □

Definition 3 (Random Serial Dictatorship (RSD))**.** Serial dictatorship with the order τ chosen uniformly at random

1	2	3
a	a	b
b	**b**	**c**
c	c	a

$1 \succ_\tau 2 \succ_\tau 3$

1	3	2
a	**b**	a
b	c	b
c	a	**c**

$1 \succ_\tau 3 \succ_\tau 2$

2	1	3
a	a	b
b	**b**	**c**
c	c	a

$2 \succ_\tau 1 \succ_\tau 3$

2	3	1
a	**b**	a
b	c	b
c	a	**c**

$2 \succ_\tau 3 \succ_\tau 1$

3	1	2
b	**a**	a
c	b	b
a	c	**c**

$3 \succ_\tau 1 \succ_\tau 2$

3	2	1
b	**a**	a
c	b	b
a	c	**c**

$3 \succ_\tau 2 \succ_\tau 1$

RSD has the following properties:

- RSD is ex post strategy-proof: regardless of the random order chosen, no player can gain from misreporting preferences. Ex post strategy-proofness is stronger than strategy-proofness.

- (Fairness) In contrast to SD, RSD satisfies equal treatment of equals. For $i, j \in N$ such that $\sigma_i = \sigma_j$, it holds that $p_{ix} = p_{jx}$ for all $x \in G$.
- RSD is ex post Pareto-efficient: every assignment in its support is Pareto efficient.

A distribution over assignments is called a **lottery**. A lottery assigns a probability (can be 0) to every possible assignment.

Definition 4 (Random Assignment)**.** A random assignment is a bistochastic matrix (each row and each column sum to 1) $P = [p_{ix}]$ where p_{ix} is the probability player i is assigned to x. For the example above, the matrix is as follows:

	a	b	c
1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$
2	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$
3	0	$\frac{2}{3}$	$\frac{1}{3}$

Definition 5 (Stochastic Dominance)**.** Random assignment P' stochastically dominates P iff for all $i \in N$ and $x \in G$,

$$\sum_{y \succeq_{\sigma_i} x} p'_{iy} \geq \sum_{y \succeq_{\sigma_i} x} p_{iy}$$

with at least one strict inequality.

Definition 6 (Ordinal Efficiency)**.** A random assignment of goods (or resources) is ordinally efficient it isn't stochastically dominated by any other assignment.

Theorem 3. *RSD is not ordinally efficient*

1	2	3	4
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

Random serial dictatorship

	a	b	c	d
1	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$
2	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{1}{4}$
4	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{1}{4}$

Stochastically dominating assignment

	a	b	c	d
1	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
2	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
3	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
4	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Agent 1 gets b in 2/24 orderings: $\{2, 1, 3, 4\}$ and $\{2, 1, 4, 3\}$. Symmetrically, agent 2 gets b in 2/24 orderings: $\{1, 2, 3, 4\}$ and $\{1, 2, 4, 3\}$; agent 3 gets a in 2/24 orderings: $\{4, 3, 2, 1\}$ and $\{4, 3, 1, 2\}$; agent 4 gets a in 2/24 orderings: $\{3, 4, 2, 1\}$ and $\{3, 4, 1, 2\}$. Now imagine the agents make a deal. If 1 or 2 have a chance of getting b, they know 3 and 4 have not yet picked yet. So, they instead take c and allow one of 3 or 4 to take b. Similarly, if 3 or 4 have the chance of taking a, they instead take c and allow 1 or 2 to take c. So 1 and 2 would not pick b and 3 and 4 would not pick a, thus shifting the probabilities to stochastically dominating assignment (right table). We note that in this stochastically dominating assignment, everyone has a strictly greater probability of getting their most preferred item, the same probability of getting their top 2 most preferred items, and the same probability of getting their top 3.

Definition 7 (Probabilistic Serial (PS) Rule)**.** A random assignment mechanism that works as such:

- Agents simultaneously "eat" their favorite goods. Each agent consumes their most preferred available item at a uniform rate (e.g., 1 unit per second). If multiple agents prefer the same item, they all consume it proportionally.
- When an item is fully consumed, agents who were consuming it move to their next preferred item. This process continues until all items are fully assigned.
- The outcome is a random assignment.

1	2	3	4
a	b	b	b
b	c	c	d
c	d	d	c
d	a	a	a

The allocation above says that player 1 has probability 1 of getting good a; player 2 has probability 1/3 of getting good b, 1/2 of getting good c, 1/6 of getting good d; player 3 has probability 1/3 of getting good b, 1/2 of getting good c, 1/6 of getting good d; and player 4 has probability 1/3 of getting good b and 2/3 of getting good d.

However, we need to derive a lottery from this random assignment.

Definition 8 (Permutation Matrix)**.** A permutation matrix is a bistochastic matrix consisting of only zeros and ones that represents an assignment.

Theorem 4 (Birkhoff-von Neumann)**.** *: Any bistochastic matrix can be obtained as a convex combination of permutation matrices*

$$\begin{array}{c|ccc} & a & b & c \\ \hline 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 2 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & 0 & \frac{1}{3} & \frac{2}{3} \end{array} = \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 \end{array} \times \frac{1}{6} + \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array} \times \frac{1}{2} + \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \end{array} \times \frac{1}{3}$$

Each permutation matrix is an assignment, and the coefficients give us the distribution. Hence we can go from any random assignment to a lottery.

Theorem 5. *Probabilistic serial is ordinally efficient.*

Theorem 6. *Probabilistic serial is not strategy-proof.*

Example 1. In the example below, player 1 can misreport their preferences to obtain a better outcome from PS.

1	2	3	4
a	a	b	b
b	c	c	c
c	d	d	d
d	b	a	a

→ **PS**

	a	b	c	d
1	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
2	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
3	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
4	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

1	2	3	4
b	a	b	b
a	c	c	c
c	d	d	d
d	b	a	a

→ **PS**

	a	b	c	d
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{4}$
2	$\frac{2}{3}$	0	$\frac{1}{12}$	$\frac{1}{4}$
3	0	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{4}$
4	0	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{4}$

Theorem 7. *There is no rule that satisfies ordinal efficiency, strategyproofness and equal treatment of equals.*

If we accept equal treatment of equals as non-negotiable then the tradeoff between ordinal efficiency and strategyproofness is unavoidable. RSD sacrifices ordinal efficiency for strategyproofness, and PS sacrifices strategyproofness for ordinal efficiency. Which do we prefer?

Pathak [2006] ran RSD and PS on truthful data from 8255 students in NYC for school assignments. The distribution over preferences for assigned schools was almost identical, despite the two mechanisms being very different. Che and Kojima [2010] show that two random assignments converge to the same limit as the instance grows larger!