CS 1360 Spring 2025
Final Exam (Practice)
— Solutions —

Problem 1: Social Choice

1. **[5 pts]** Define (using mathematical notation) the concept of a *neutral* social choice function.

   Solution: Let A be a finite set of alternatives, $N = \{1, \ldots, n\}$ the set of voters, and $\mathcal{L}(A)$ the set of all strict orders on A. A social choice function is $f : \mathcal{L}(A)^N \to A$. It is neutral if for every bijection $\pi : A \to A$ and every profile $\boldsymbol{\sigma} \in \mathcal{L}(A)^N$, $f\big(\pi(\boldsymbol{\sigma})\big) = \pi\big(f(\boldsymbol{\sigma})\big)$, where $\pi(\boldsymbol{\sigma}) = (\pi(\sigma_1), \ldots, \pi(\sigma_n))$.

2. **[15 pts]** A social choice function f is *anonymous* if for any permutation $\pi : N \to N$ and any preference profile $\boldsymbol{\sigma}$, $f(\sigma_1, \sigma_2, \ldots, \sigma_n) = f(\sigma_{\pi(1)}, \sigma_{\pi(2)}, \ldots, \sigma_{\pi(n)})$. Informally, changing the order (or "names") of voters does not change the outcome. Prove that, for the case of two voters and two alternatives, no social choice function is both anonymous and neutral.

   Solution: Let $A = \{a, b\}$ and $N = \{1, 2\}$. Assume for contradiction that f is anonymous and neutral. Consider the profile

   $$\boldsymbol{\sigma} = (\sigma_1, \sigma_2) \quad \text{with} \quad \sigma_1 : a \succ b, \sigma_2 : b \succ a.$$

   Assume without loss of generality that $f(\boldsymbol{\sigma}) = a$. Now let $\tau : N \to N$ be the transposition swapping voters 1 and 2, and let $\pi : A \to A$ be the transposition swapping alternatives a and b. Note that applying either π or τ to $\boldsymbol{\sigma}$ leads to the same profile $\boldsymbol{\sigma}'$. By anonymity, $f(\boldsymbol{\sigma}') = f(\tau(\boldsymbol{\sigma})) = a$. But by neutrality, $f(\boldsymbol{\sigma}') = f(\pi(\boldsymbol{\sigma})) = \pi(a) = b$. This is a contradiction since $a \neq b$.

Problem 2: Indivisible Goods

1. **[5 pts]** Define (using mathematical notation) the notion of *envy freeness up to one good* (EF1).

   Solution: We say the allocation A is EF1 if

   $$\forall i, j \in N,\ \exists g \in A_j \quad \text{such that} \quad V_i(A_i) \geq V_i\big(A_j \setminus \{g\}\big).$$

2. **[15 pts]** Let there be two players with additive, strictly positive valuations over a set of goods. Consider the following algorithm: player 1 divides the goods into two bundles X_1 and X_2 in a way that maximizes $\min\{V_1(X_1), V_1(X_2)\}$ (intuitively, player 1 divides the goods as evenly as possible according to their own valuation). Then, player 2 chooses their favorite bundle, and player 1 receives the remaining bundle. Prove that this algorithm produces an EF1 allocation.

   Solution: Let the resulting allocation be (A_1, A_2) where $A_2 = X_k, A_1 = X_{3-k}$.

   (i) No envy by player 2: By choice, $V_2(A_2) = V_2(X_k) \geq V_2(X_{3-k}) = V_2(A_1)$, so player 2 envies player 1 by at most zero goods (hence EF1 holds trivially for $i = 2$).

   (ii) EF1 for player 1: Since player 1 chose (X_1, X_2) to maximize $\min\{V_1(X_1), V_1(X_2)\}$, it follows that

   $$\big|V_1(X_1) - V_1(X_2)\big| \;\leq\; \max_{g \in A} V_1(\{g\}), \tag{1}$$

   because otherwise moving the single most-valuable good from the richer bundle to the poorer would increase the minimum. In particular, if $A_2 = X_k$ is the bundle player 2 picks, then

   $$V_1(A_1) = V_1(X_{3-k}) \;\geq\; V_1(X_k) \;-\; \max_{g \in A} V_1(\{g\}) = V_1(A_2) \;-\; \max_{g \in A_2} V_1(\{g\}).$$

   Hence there exists some good $g \in A_2$ (namely the one of maximum V_1-value) such that

   $$V_1(A_1) \;\geq\; V_1\big(A_2 \setminus \{g\}\big),$$

   which is precisely EF1 for $i = 1$.

   Combining (i) and (ii) shows the allocation is EF1.

   Note: The same argument actually shows the stronger property of EFX, since Equation (1) holds even with min instead of max.

Problem 3: Online Matching Algorithms

1. **[5 pts]** Define (using mathematical notation) the competitive ratio of an online (bipartite) matching algorithm.

   Solution: ALG has competitive ration $\alpha \leq 1$ if for every graph $G = (U, V, E)$ and every input order π of V, $ALG(G, \pi)/OPT(G) \geq \alpha$.

2. **[15 pts]** Consider the following online matching algorithm. When an online vertex $v \in V$ arrives, if it has an offline neighbor u that has already been matched (that is, there is $u \in U$ such that $(u, v) \in E$ and u has been matched with $v' \in V$ that arrived before v), don't match v. Otherwise, match v with an arbitrary unmatched neighbor.

What is the competitive ratio of this algorithm, as a function of n (the number of vertices on each side)?

Note: Establish an upper bound and a lower bound. The two bounds should ideally be equal.

Solution: We claim the competitive ratio is exactly $1/n$.

Upper bound. Take the complete bipartite graph $K_{n,n}$ with any arrival order. On the very first arrival v_1, no offline vertex is yet matched, so $\mathcal{A}$ matches it to some u_1. But every subsequent v_i sees that u_1 is already matched, so $\mathcal{A}$ leaves all $v_2, \ldots, v_n$ unmatched. Hence

$$ALG(\sigma) = 1, OPT(\sigma) = n \Longrightarrow \frac{ALG(\sigma)}{OPT(\sigma)} = \frac{1}{n}.$$

Lower bound. On any input σ with $OPT(\sigma) = k \geq 1$, there is at least one edge $(u, v) \in E$. Let v^* be the first arriving vertex that has at least one neighbor in U. Since no offline vertex was matched before v^* (all earlier arrivals had no neighbors), the algorithm will match v^* to some free neighbor. Thus $ALG(\sigma) \geq 1$, and

$$\frac{ALG(\sigma)}{OPT(\sigma)} \geq \frac{1}{k} \geq \frac{1}{n}.$$

Problem 4: Cascade Models

1. **[5 pts]** Define the *contagion threshold* of an infinite graph (with bounded degrees).

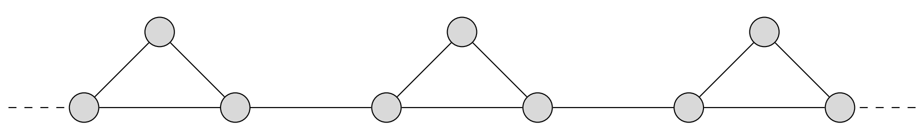
   Solution: Given a finite seed set $S \subset V$, set

   $$A_0 = S, \qquad A_{t+1} = A_t \cup \big\{ v \notin A_t : |N(v) \cap A_t| \geq \lceil q \cdot |N(v)| \rceil \big\},$$

   and write $A_\infty = \bigcup_{t \geq 0} A_t$. We say S is contagious if $A_\infty = V$. The contagion threshold of G is

   $$q_c(G) = \max\big\{ q \in [0,1] : \text{there exists a finite contagious seed } S \big\}.$$

2. **[15 pts]** Consider a graph G which is composed of an infinite sequence of triangles, where adjacent triangles are connected by a single edge, as shown below:



What is the contagion threshold of G?

Note: Establish an upper bound and a lower bound. The two bounds should ideally be equal.

Solution: We show $q_c(G) = \frac{1}{3}$.

Lower bound ($q_c \geq \frac{1}{3}$): If $q \leq \frac{1}{3}$, then for every vertex of degree $d \in \{2, 3\}$, $\lceil qd \rceil \leq 1$. Hence the activation rule reduces to "at least one active neighbor," and a single-vertex seed $S = \{v\}$ will eventually flood the entire infinite path of triangles.

Upper bound ($q_c \leq \frac{1}{3}$): Suppose by way of contradiction that $q > \frac{1}{3}$ and yet there is some finite seed set S whose activation eventually reaches every vertex. Observe that in our "chain of triangles" each bridge vertex (the one connecting two consecutive triangles) has degree 3 (two neighbors within its own triangle, plus one neighbor across the bridge). Since $q > \frac{1}{3}$, we have $\lceil q \cdot 3 \rceil = 2$, so each bridge vertex requires at least *two* active neighbors before it can become active itself.

Now any finite seed S can only occupy finitely many triangles. Let $T_{\max}$ be the right-most triangle that ever contains an active vertex. By definition of $T_{\max}$, all vertices in the next triangle to the right are initially inactive, and they remain so until the bridge vertex between $T_{\max}$ and that next triangle activates. But at the moment we attempt to activate this bridge vertex, its only active neighbor is the single bridge-edge endpoint in $T_{\max}$—the other two neighbors (its two partners inside the next triangle) are still inactive. Hence it has only one active neighbor, which is strictly less than the required two, and so it cannot become active.

Therefore the cascade cannot cross from $T_{\max}$ into the next triangle, contradicting the assumption that S was contagious. We conclude that no finite seed can ever activate the entire graph when $q > \frac{1}{3}$, and thus $q_c(G) \leq \frac{1}{3}$.

Problem 5: Feature Attribution

1. **[5 pts]** Define (using mathematical notation) the *Shapley value* $\sigma_i(N, v)$ of player $i \in N$ in a cooperative game (N, v).

   Solution: Let (N, v) be a cooperative game with $|N| = n$ and $v(\emptyset) = 0$. For each permutation π of N, write $S_\pi(i) = \{j \in N : \pi \text{ places } j \text{ before } i\}$ for the set of players preceding i in π. Then the Shapley value of player i is

$$\sigma_i(N, v) = \frac{1}{n!} \sum_{\pi} \Big[v\big(S_\pi(i) \cup \{i\}\big) - v\big(S_\pi(i)\big) \Big].$$

2. **[15 pts]** Prove that $\sum_{i \in N} \sigma_i(N, v) = v(N)$.

   Note: This was done in class.

Solution: We must show $\sum_{i\in N}\sigma_i(N,v)=v(N)$. By the permutation definition,

$$\sum_{i\in N}\sigma_i(N,v)=\sum_{i\in N}\frac{1}{n!}\sum_{\pi\in\mathfrak{S}_N}\big[v(S_\pi(i)\cup\{i\})-v(S_\pi(i))\big].$$

Interchange the sums over i and over permutations π:

$$\sum_{i\in N}\sigma_i(N,v)=\frac{1}{n!}\sum_{\pi\in\mathfrak{S}_N}\sum_{i\in N}\big[v(S_\pi(i)\cup\{i\})-v(S_\pi(i))\big].$$

Now, for each fixed permutation $\pi=(\pi(1),\dots,\pi(n))$, the inner sum $\sum_{i\in N}\big[v(S_\pi(i)\cup\{i\})-v(S_\pi(i))\big]$ is exactly the telescoping sum of marginal contributions as players enter in the order π:

$$\big[v(\{\pi(1)\})-v(\emptyset)\big]+\big[v(\{\pi(1),\pi(2)\})-v(\{\pi(1)\})\big]+\cdots+\big[v(N)-v(N\setminus\{\pi(n)\})\big].$$

All intermediate terms cancel, leaving

$$\sum_{i\in N}\big[v(S_\pi(i)\cup\{i\})-v(S_\pi(i))\big]=v(N)-v(\emptyset)=v(N).$$

Since this holds for every π, we get $\sum_{i\in N}\sigma_i(N,v)=\frac{1}{n!}\sum_{\pi\in\mathfrak{S}_N}v(N)=v(N)$. Thus the Shapley value vector distributes the entire worth $v(N)$ among the players, proving efficiency.