Economics and Computation (Spring 2025)
Assignment #2

Due: 2/26/2025 11:59pm ET

Problem 1: The price of anarchy

Consider the following scheduling game. The players $N = \{1, \ldots, n\}$ are associated with tasks, each with weight w_i. There is also a set M of m machines. Each player chooses a machine to place their task on, that is, the strategy space of each player is M. A strategy profile induces an assignment $A : N \to M$ of players (or tasks) to machines; the *cost* of player i is the total load on the machine to which i is assigned: $\ell_{A(i)} = \sum_{j \in N:\ A(j)=A(i)} w_j$. Our objective function is the *makespan*, which is the maximum load on any machine: $\text{cost}(A) = \max_{\mu \in M} \ell_\mu$. It is known that scheduling games always have pure Nash equilibria.

1. **[15 points]** Let G be a scheduling game with n tasks of weight $w_1, \ldots, w_n$, and m machines. Let $A : N \to M$ be a Nash equilibrium assignment. Prove that

$$\text{cost}(A) \leq \left(2 - \frac{2}{m+1}\right) \cdot \text{opt}(G).$$

   That is, the price of anarchy is at most $2 - 2/(m+1)$.

2. **[10 points]** Prove that the upper bound of part (a) is tight, by constructing an appropriate family of scheduling games for each $m \in \mathbb{N}$.

Problem 2: Voting rules

[10 points] When the number of alternatives is m, a *positional scoring rule* is defined by a score vector $(s_1, \ldots, s_m)$ such that $s_k \geq s_{k+1}$ for all $k = 1, \ldots, m-1$. Each voter gives s_k points to the alternative they rank in position k, and the points are summed over all voters. We discussed two examples of positional scoring rules: plurality, defined by the vector $(1, 0, \ldots, 0)$, and Borda, defined by the vector $(m-1, m-2, \ldots, 0)$. Another common example is *veto*, defined by the vector $(1, \ldots, 1, 0)$.

For the case of $m = 3$, prove that any positional scoring vector with $s_2 > s_3$ is *not* Condorcet consistent.

Hint: It is possible to do this via a single preference profile that includes 7 voters.

Problem 3: The epistemic approach to voting

[10 points] Suppose that there is a true ranking of m alternatives, each of n voters evaluates all pairs of alternatives according to the Condorcet noise model (Lecture 6, slide 5) with $p > 1/2$, and these comparisons are aggregated into a voting matrix. Prove that the output of the Kemeny rule applied to this voting matrix coincides with the true ranking with probability that goes to 1 as n goes to infinity.

Hint: Use the Condorcet Jury Theorem (or the law of large numbers).

Problem 4: Strategic manipulation in elections

We saw in class a proof sketch of the Gibbard-Satterthwaite Theorem for the special case of strategyproof and neutral voting rules with $m \geq 3$ and $m \geq n$. That proof relied on two key lemmas. In this problem, you will prove the two lemmas and formalize the theorem's proof for this special case.

Prove the following statements.

1. **[10 points]** Let f be a strategyproof voting rule, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be a preference profile, and $f(\boldsymbol{\sigma}) = a$. If $\boldsymbol{\sigma'}$ is a profile such that $[a \succ_{\sigma_i} x \;\Rightarrow\; a \succ_{\sigma'_i} x]$ for all $x \in A$ and $i \in N$, then $f(\boldsymbol{\sigma'}) = a$.

2. **[10 points]** Let f be a strategyproof and onto voting rule. Furthermore, let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be a preference profile and $a, b \in A$ such that $a \succ_{\sigma_i} b$ for all $i \in N$. Then $f(\boldsymbol{\sigma}) \neq b$.

   Hint: use part (a).

3. **[10 points]** Let m be the number of alternatives and n be the number of voters, and assume that $m \geq 3$ and $m \geq n$. Furthermore, let f be a strategyproof and neutral voting rule. Then f is dictatorial.

   Important note: There are many proofs of the full version of the Gibbard-Satterthwaite Theorem; *here the task is specifically to formalize the proof sketch we did in class.*

Problem 5: Formulate a Research Question

[25 points] Formulate a research question that is relevant to one of the topics covered in this assignment: the price of anarchy, voting rules, the epistemic approach to voting, and strategic manipulation in elections. Refer to this document for guidelines.

During the process of formulating your question, keep track of your findings in a "research journal." At a minimum, it should include brainstorming ideas for questions and notes on relevant papers that you have identified.

Please submit the following deliverables:

1. Your research question.
2. A brief explanation of why it satisfies each of the following criteria:
   (a) Relevant: Which course topics is the question related to?

(b) Nontrivial: What is an immediate way of attempting to answer the question and why does it fail?

(c) Feasible: How would you tackle the question if you had the entire semester?

Note: This item is (still) optional in Assignment #2.

(d) Novel: List the 1–3 most closely related papers that you have identified in your literature review and explain how your question differs.

3. Append your research journal to the PDF that contains your solutions. The research journal will not be graded; it is there to show your work.