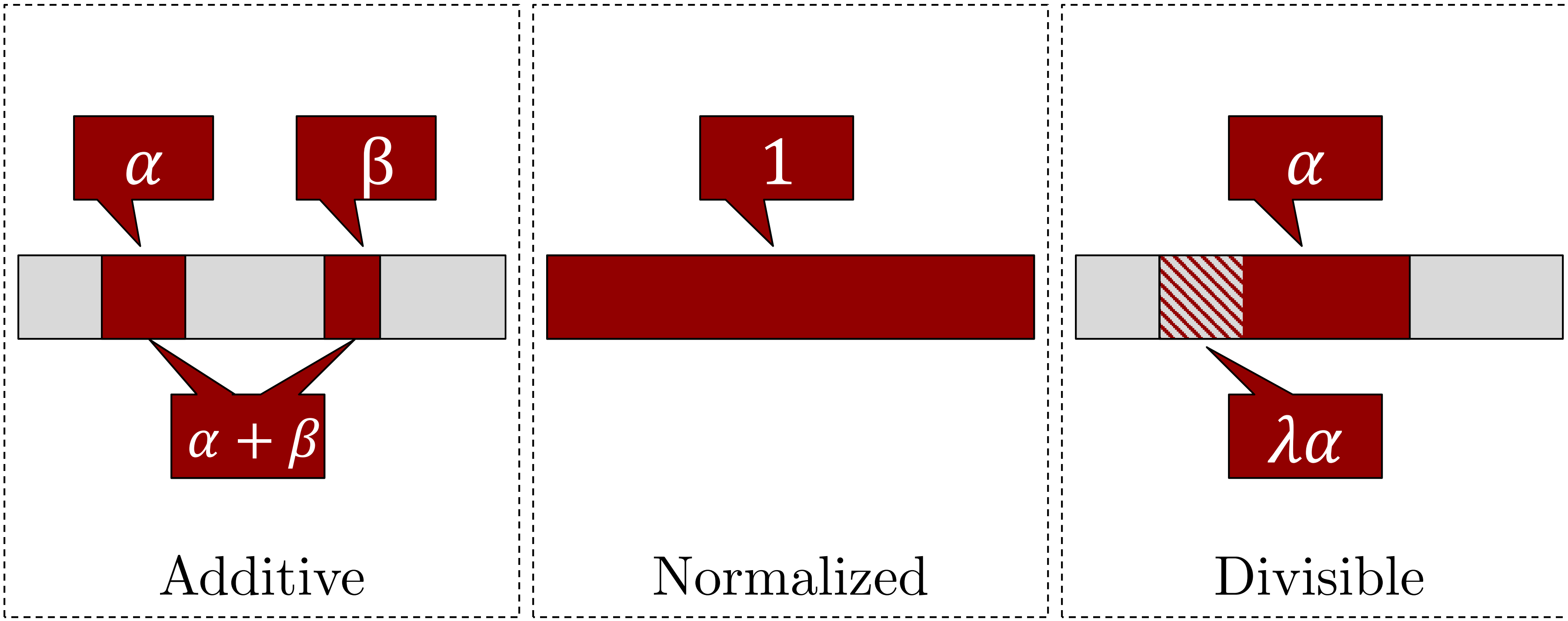
CMU 15-896

Fair division 1: Cake cutting

Teacher:
Ariel Procaccia

- Single heterogeneous good, represented as $[0,1]$
- Set of players $N = \{1, \ldots, n\}$
- Piece of cake $X \subseteq [0,1]$: finite union of disjoint intervals

Each player i has a valuation V_i that is:



Fairness, formalized

- Our goal is to find an allocation $A_1, \dots, A_n$
- Proportionality:

$$\forall i \in N, V_i(A_i) \geq \frac{1}{n}$$

- Envy-Freeness (EF):

$$\forall i, j \in N, V_i(A_i) \geq V_i(A_j)$$

Fairness, formalized

Poll 1: What is the relation between proportionality and EF?

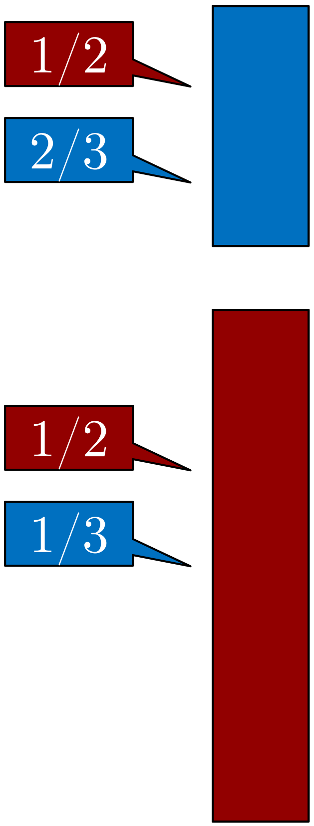
1. Proportionality ⇒ EF
2. EF ⇒ proportionality
3. Equivalent
4. Incomparable

Cut-and-Choose

- Algorithm for $n = 2$ [Procaccia and Procaccia, circa 1985]
- Player 1 divides into two pieces X, Y s.t.

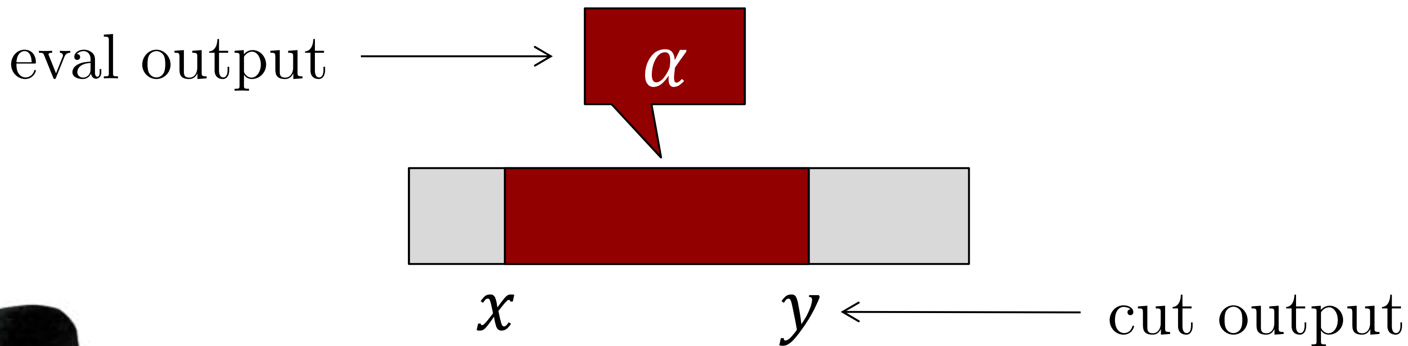
$$V_1(X) = 1/2, V_1(Y) = 1/2$$

- Player 2 chooses preferred piece
- This is EF and proportional



The Robertson-Webb model

- What is the time complexity of C&C?
- Input size is n
- Two types of queries
  - $\text{Eval}_i(x, y)$ returns $V_i([x, y])$
  - $\text{Cut}_i(x, \alpha)$ returns y such that $V_i([x, y]) = \alpha$



The Robertson-Webb model

- Two types of queries
  - $\text{Eval}_i(x, y) = V_i([x, y])$
  - $\text{Cut}_i(x, \alpha) = y$ s.t. $V_i([x, y]) = \alpha$

#queries needed to find an EF allocation when $n = 2$?

Dubins-Spanier

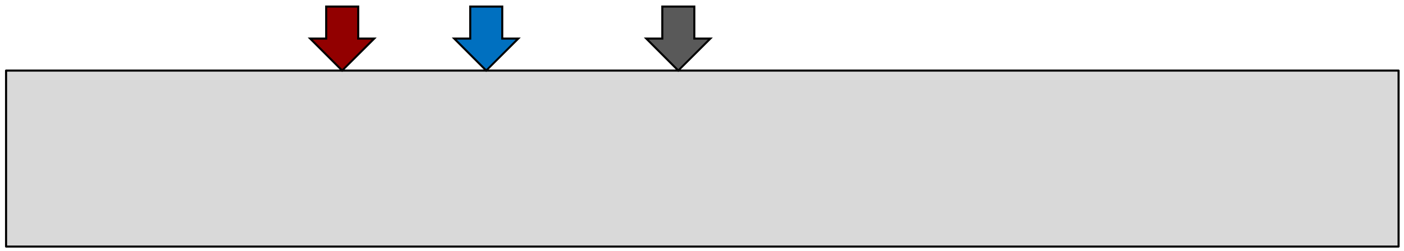
- Referee continuously moves knife
- Repeat: when piece left of knife is worth $1/n$ to player, player shouts "stop" and gets piece
- That player is removed
- Last player gets remaining piece

Dubins-Spanier

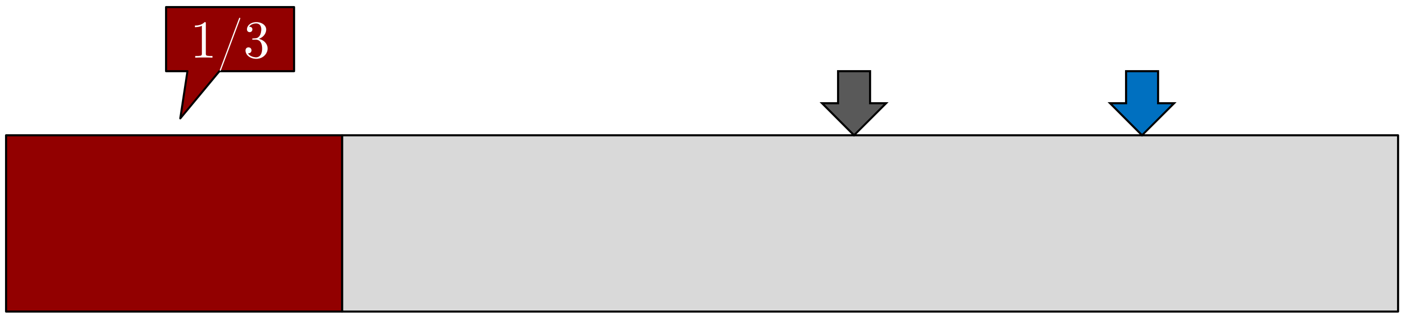
Poll 2: What is the complexity of DS in the RW model?

1. $\Theta(n)$
2. $\Theta(n \log n)$
3. $\Theta(n^2)$
4. $\Theta(n^2 \log n)$

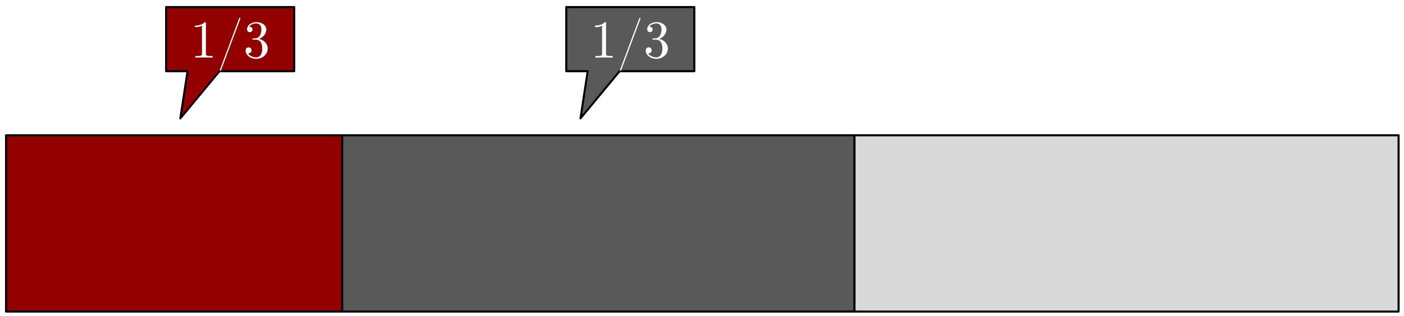
Dubins-Spanier



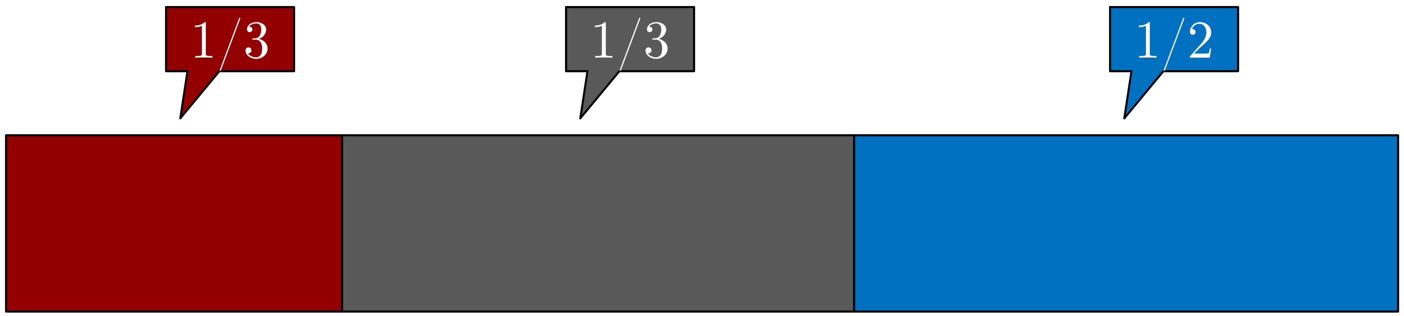
Dubins-Spanier



Dubins-Spanier



Dubins-Spanier



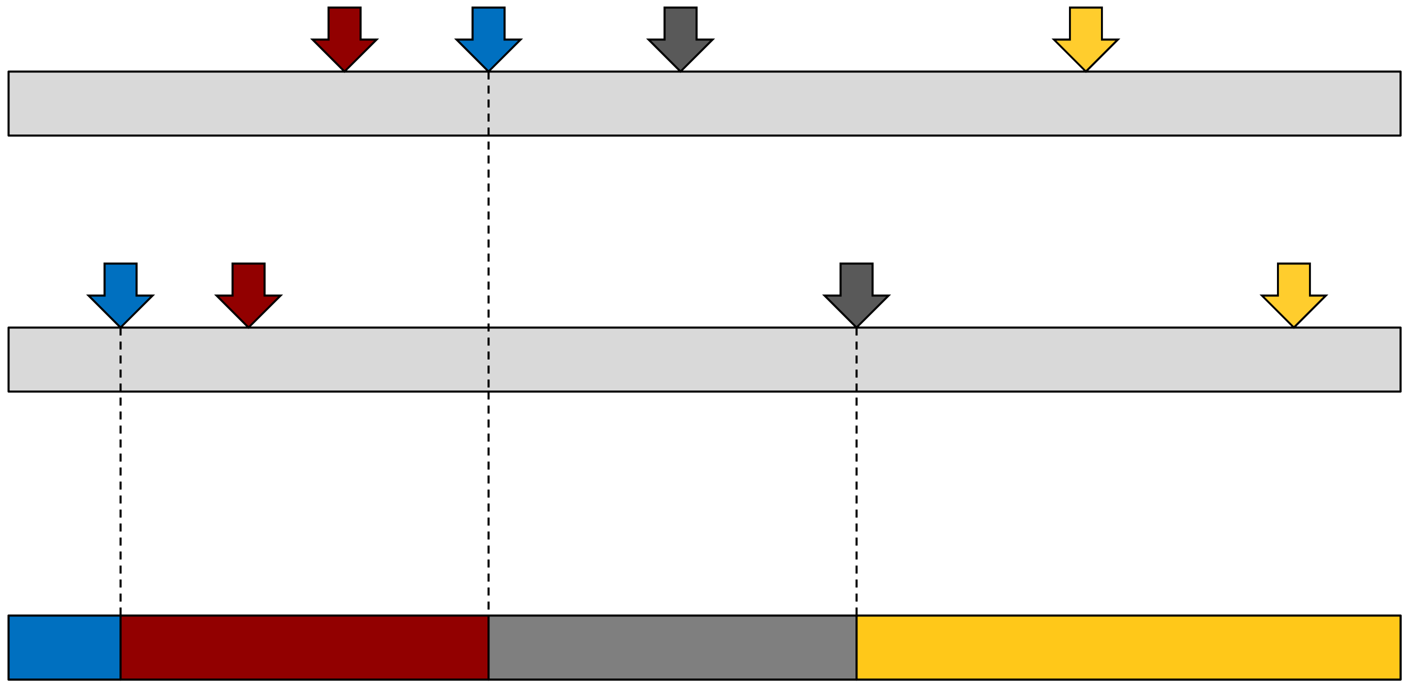
Even-Paz

- Given $[x, y]$, assume $n = 2^k$
- If $n = 1$, give $[x, y]$ to the single player
- Otherwise, each player i makes a mark z s.t.

$$V_i([x, z]) = \frac{1}{2} V_i([x, y])$$

- Let z^* be the $n/2$ mark from the left
- Recurse on $[x, z^*]$ with the left $n/2$ players, and on $[z^*, y]$ with the right $n/2$ players

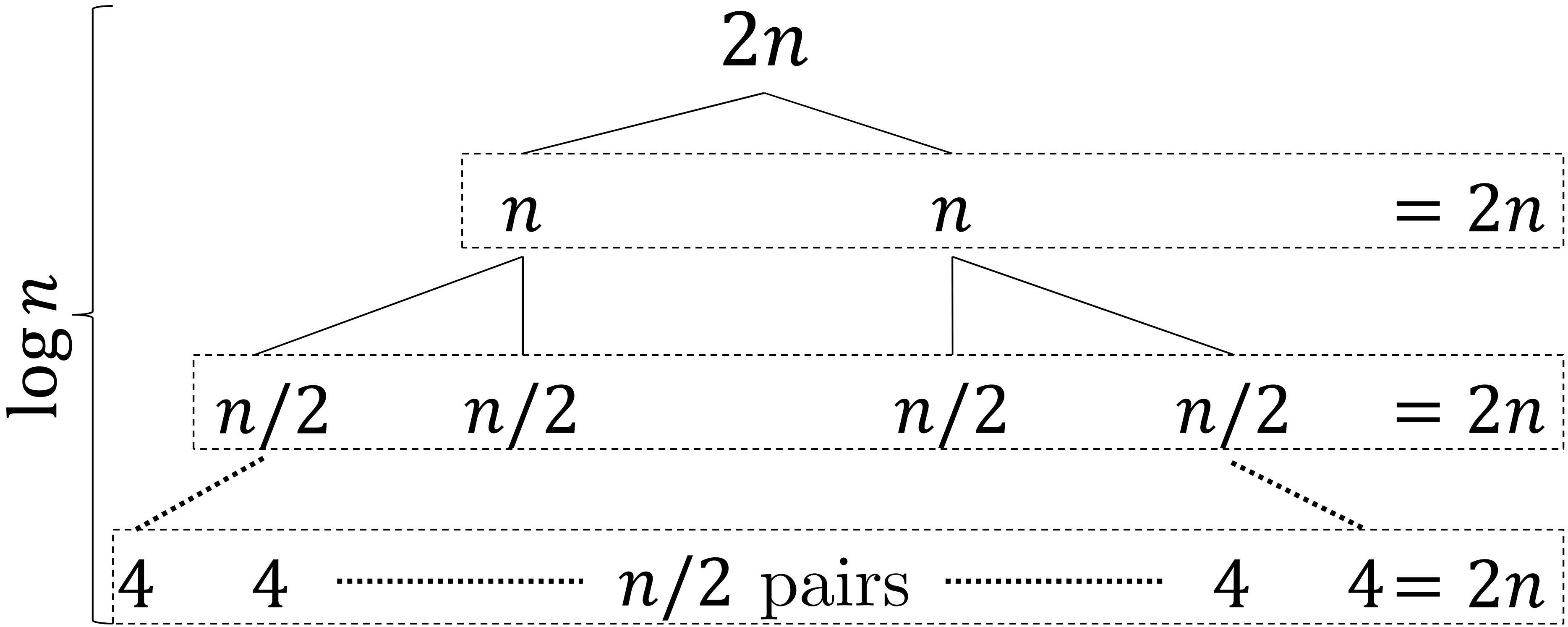
Even-Paz



Even-Paz: propotionality

- **Claim**: The Even-Paz protocol produces a proportional allocation
- **Proof:**
    - At stage 0, each of the n players values the whole cake at 1
    - At each stage the players who share a piece of cake value it at least at $V_i([x,y])/2$
    - Hence, if at stage k each player has value at least $1/2^k$ for the piece he's sharing, then at stage $k+1$ each player has value at least $\frac{1}{2^{k+1}}$
    - The number of stages is $\log n$ ■

$$T(1) = 0, T(n) = 2n + 2T\left(\frac{n}{2}\right)$$

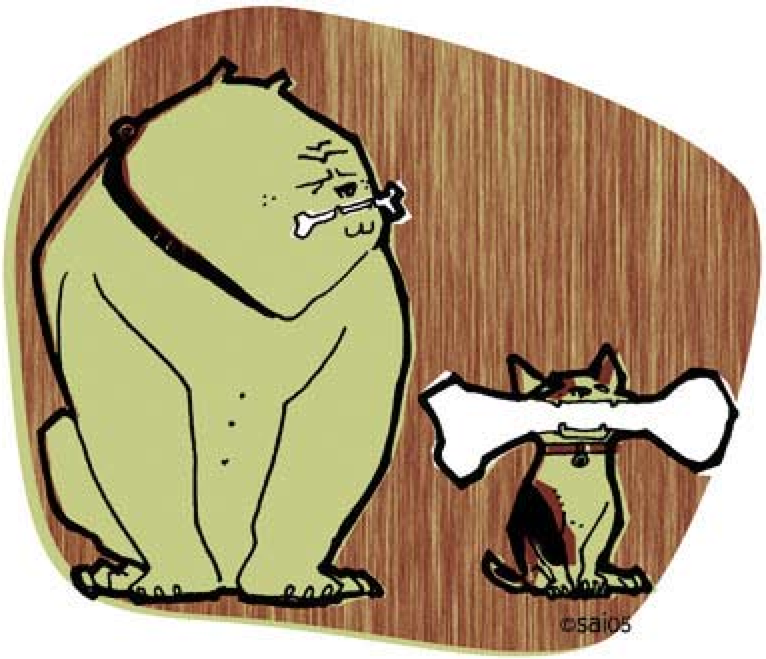


Overall: $2n \log n$

Complexity of proportionality

- Theorem [Edmonds and Pruhs 2006]: Any proportional protocol needs $\Omega(n \log n)$ operations in the RW model
- We will prove the theorem on Wednesday
- The Even-Paz protocol is provably optimal!

What about envy?



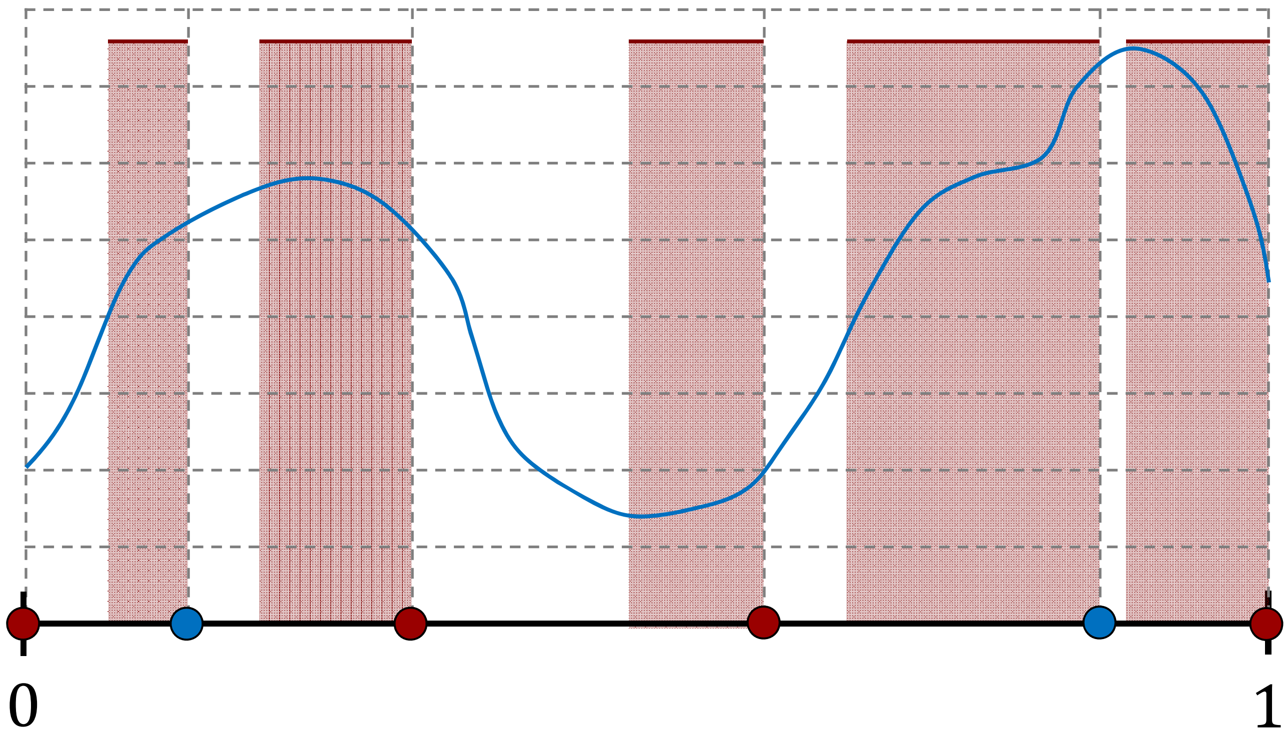
Selfridge-Conway

- **Stage 0**
  - Player 1 divides the cake into three equal pieces according to V_1
  - Player 2 trims the largest piece s.t. there is a tie between the two largest pieces according to V_2
  - Cake 1 = cake w/o trimmings, Cake 2 = trimmings
- **Stage 1 (division of Cake 1)**
  - Player 3 chooses one of the three pieces of Cake 1
  - If player 3 did not choose the trimmed piece, player 2 is allocated the trimmed piece
  - Otherwise, player 2 chooses one of the two remaining pieces
  - Player 1 gets the remaining piece
  - Denote the player $i \in \{2,3\}$ that received the trimmed piece by T, and the other by T'
- **Stage 2 (division of Cake 2)**
  - T' divides Cake 2 into three equal pieces according to $V_{T'}$
  - Players T, 1, and T' choose the pieces of Cake 2, in that order

The complexity of EF

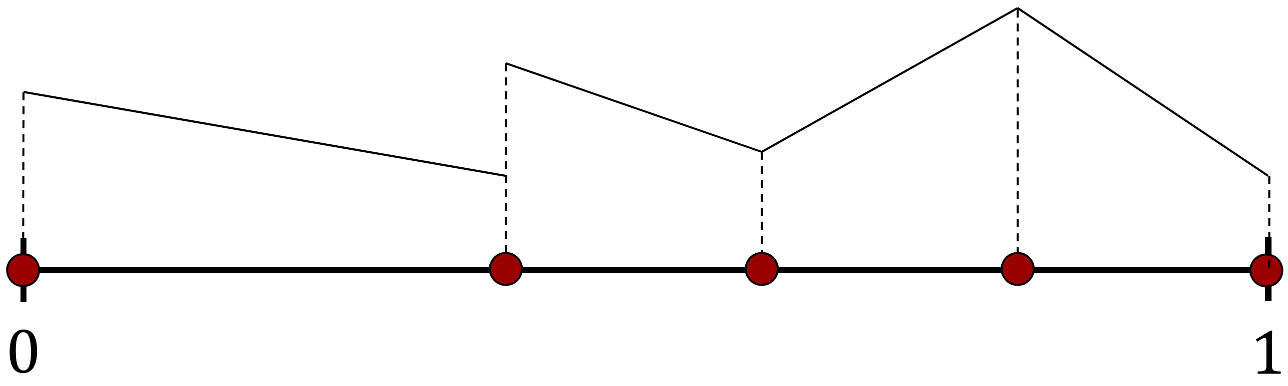
- Theorem [Brams and Taylor 1995]: There is an unbounded EF cake cutting algorithm in the RW model
- Theorem [P 2009]: Any EF algorithm requires $\Omega(n^2)$ queries in the RW model
- Theorem [Kurokawa et al. 2013]: EF cake cutting with piecewise uniform valuations is as hard as general case

The complexity of EF



The complexity of EF

- Theorem [Kurokawa et al. 2013]: EF cake cutting with piecewise linear valuations is polynomial in the number of breakpoints



A subtlety

- EF protocol that uses n queries
- f = 1-1 mapping from valuation functions to [0,1]
- The protocol asks each player $\mathbf{cut}_i(0, 1/2)$
- Player i replies with $y_i = f(V_i)$
- The protocol computes $V_i = f^{-1}(y_i)$
- Is this a valid EF protocol in the RW model?

Strategyproof Cake cutting

- All the cake cutting algorithms we discussed are not SP: agents can gain from manipulation
  - Cut and choose: player 1 can manipulate
  - Dubins-Spanier: shout later
- Assumption: agents report full valuations
- Deterministic EF and SP algs exist in some special cases, but they are rather involved [Chen et al. 2010]

A randomized algorithm

- $X_1, \dots, X_n$ is a perfect partition if $V_i(X_j) = 1/n$ for all i, j
- Algorithm
  - Compute a perfect partition
  - Draw a random permutation π over $\{1, \dots, n\}$
  - Allocate to agent i the piece $X_{\pi(i)}$
- Theorem [Chen et al. 2010; Mossel and Tamuz 2010]: the algorithm is SP in expectation and always produces an EF allocation
- Proof: if an agent lies the algorithm may compute a different partition, but for any partition:

$$\sum_{j \in N} \frac{1}{n} V_i(X'_j) = \frac{1}{n} \sum_{j \in N} V_i(X'_j) = \frac{1}{n} \quad \blacksquare$$

Computing a perfect partition

- Theorem [Alon, 1986]: a perfect partition always exists, needs polynomially many cuts
- Proof is nonconstructive
- Can find perfect partitions for special valuation functions