CMU 15-896

Social choice 5: Voting Rules as MLEs

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Condorcet strikes again

- For Condorcet [1785], the purpose of voting is not merely to balance subjective opinions; it is a collective quest for the truth
- Enlightened voters try to judge which alternative best serves society
- For $m = 2$ the majority opinion will very likely be correct
- Realistic in trials by jury or the pooling of expert opinions — or in human computation!

Motivation: EteRNA

- Developed at CMU (Adrien Treuille) and Stanford
- Choose 8 RNA designs to synthesize
- Some designs are truly more stable than others
- The goal of voting is to compare the alternatives by true quality

Condorcet's noise model

- True ranking of the alternatives
- Voting pairwise on alternatives, each comparison is correct with prob. $p > 1/2$
- Results are tallied in a voting matrix

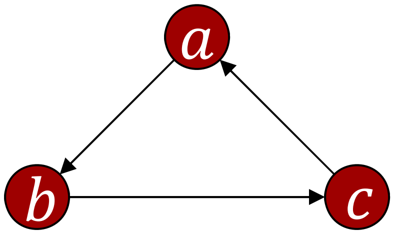
	a	b	c
a	-	8	6
b	5	-	11
c	7	2	-

What is the Borda score of alternative b?

Condorcet's 'solution'

- Condorcet's goal: find "the most probable" ranking
- Condorcet suggested: take the majority opinion for each comparison; if a cycle forms, "successively delete the comparisons that have the least plurality"
- In example, we delete $c \succ a$ to get $a \succ b \succ c$

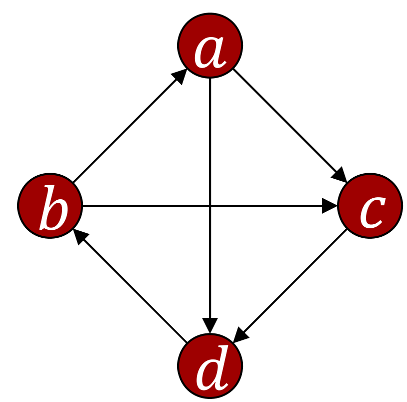
	a	b	c
a	-	8	6
b	5	-	11
c	7	2	-



Condorcet's 'solution'

- With four alternatives we get ambiguities
- In example, order of strength is $c \succ d$, $a \succ d$, $b \succ c$, $a \succ c$, $d \succ b$, $b \succ a$
- Delete $b \succ a \Rightarrow$ still cycle
- Delete $d \succ b \Rightarrow$ either a or b could be top-ranked

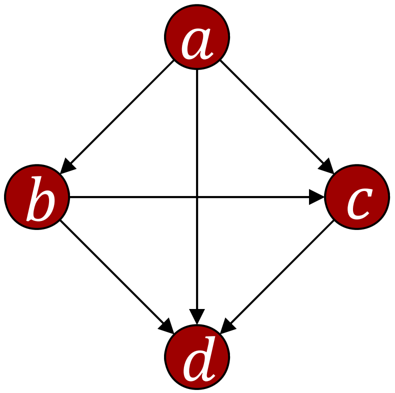
	a	b	c	d
a	-	12	15	17
b	13	-	16	11
c	10	9	-	18
d	8	14	7	-



Condorcet's 'solution'

- Did Condorcet mean we should reverse the weakest comparisons?
- Reverse $b \succ a$ and $d \succ b \Rightarrow$ we get $a \succ b \succ c \succ d$, with 89 votes
- $b \succ a \succ c \succ d$ has 90 votes (only reverse $d \succ b$)

	a	b	c	d
a	-	12	15	17
b	13	-	16	11
c	10	9	-	18
d	8	14	7	-



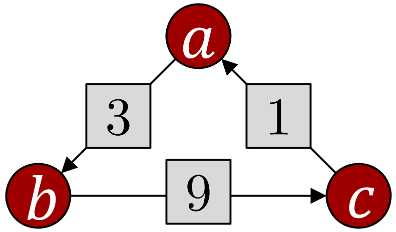
Exasperation?

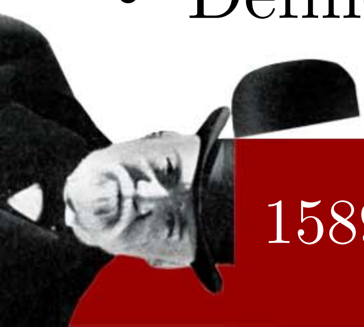
- "The general rules for the case of any number of candidates as given by Condorcet are stated so briefly as to be hardly intelligible . . . and as no examples are given it is quite hopeless to find out what Condorcet meant" [Black 1958]
- "The obscurity and self-contradiction are without any parallel, so far as our experience of mathematical works extends ...

Young's solution

- M = matrix of votes
- Suppose true ranking is $a \succ b \succ c$; prob of observations $\Pr[M \mid \succ]$:
  $\binom{13}{8} p^8(1-p)^5 \cdot \binom{13}{6} p^6(1-p)^7 \cdot \binom{13}{11} p^{11}(1-p)^2$
- For $a \succ c \succ b$, $\Pr[M \mid \succ]$ is
  $\binom{13}{8} p^8(1-p)^5 \cdot \binom{13}{6} p^6(1-p)^7 \cdot \binom{13}{2} p^2(1-p)^{11}$
- Coefficients are identical
- Define the pairwise comparison graph

	a	b	c
a	-	8	6
b	5	-	11
c	7	2	-

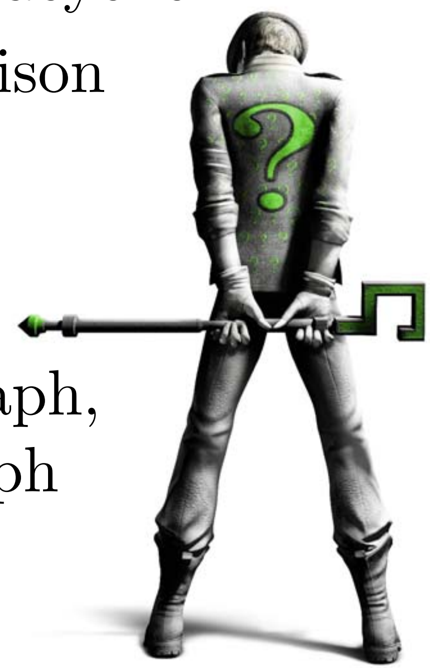




Young's solution

Poll 1: Which ranking $\succ$ maximizes $\Pr[M \mid \succ]$?

1. Delete edges in the pairwise comparison graph according to weight, until it becomes acyclic
2. Reverse edges in the pairwise comparison graph according to weight, until it becomes acyclic
3. Find a set of edges of minimum total weight in the pairwise comparison graph, such that if they are reversed the graph becomes acyclic
4. Apply Borda count to M



Young's solution

- $\Pr[\succ|M] = \frac{\Pr[M|\succ] \cdot \Pr[\succ]}{\Pr[M]}$
- Assume uniform prior over $\succ$, $\Pr[\succ] = \frac{1}{m!}$
- Must maximize $\Pr[M|\succ]$
- The optimal rule maximizes #agreements with voters on pairs of candidates
- This rule is called the Kemeny rule

The Kemeny rule

- Theorem [Bartholdi, Tovey, Trick 1989]: Computing the Kemeny ranking is NP-hard
- Typically formulated as an ILP: for every $(a, b) \in A^2$, $x_{(a,b)} = 1$ iff a is ranked above b, and

$$w_{(a,b)} = |\{i \in N \mid a \succ_i b\}|$$

The Kemeny rule

Maximize $\sum_{(a,b)} x_{(a,b)} w_{(a,b)}$

Subject to

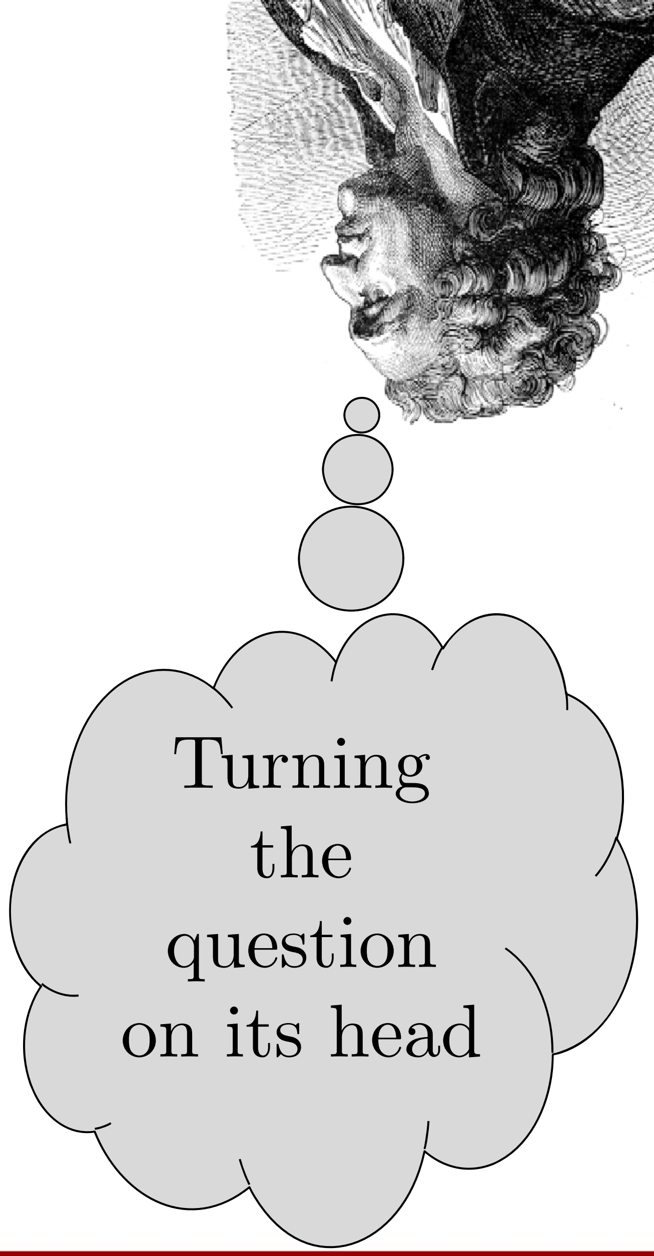
For all distinct $a, b \in A, x_{(a,b)} + x_{(b,a)} = 1$

For all distinct $a, b, c \in A, x_{(a,b)} + x_{(b,c)} + x_{(c,a)} \leq 2$

For all distinct $a, b \in A, x_{(a,b)} \in \{0,1\}$

Ten years later...

- Noise model = distribution over votes (rankings) for each true ranking
- Votes are drawn independently
- Which voting rules have a noise model for which they are MLEs of the true ranking?



Scoring rules as MLEWs

- Theorem [Conitzer and Sandholm 2005]: Any scoring rule is an MLE
- Proof:
  - $x_1 \succ^* x_2 \succ^* \cdots \succ^* x_m$ = true ranking
  - The probability that a voter i ranks each alternative x_j in position r_{ij} is prop. to

$$\prod_{j=1}^{m} (m+1-j)^{s_{r_{ij}}}$$

Scoring rules as MLEWs

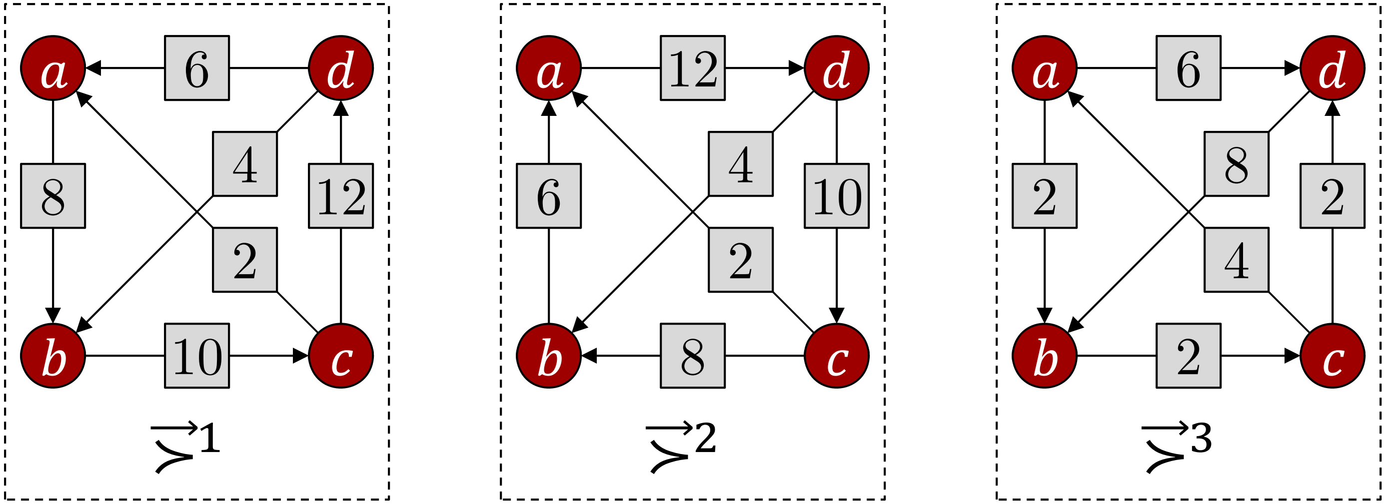
- Proof (continued):
  - $\Pr[M \mid \prec^*] \propto \prod_{i=1}^{n} \prod_{j=1}^{m} (m+1-j)^{s_{r_{ij}}}$
  - This is equal to
  $$\prod_{j=1}^{m} (m+1-j)^{\sum_{i=1}^{n} s_{r_{ij}}}$$
  - $m+1-j$ is positive and decreasing in j, so to maximize label alternative with kth highest score as x_k ■

Maximin is not an MLEW

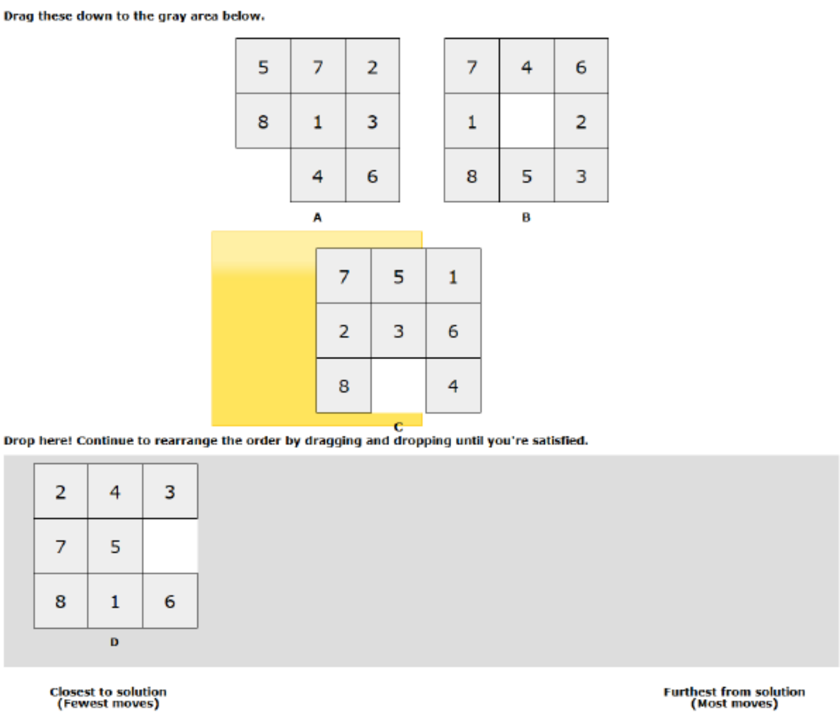
- Lemma: If there exist preference profiles $\overrightarrow{\succ}^1$ and $\overrightarrow{\succ}^2$ such that $f(\overrightarrow{\succ}^1) = f(\overrightarrow{\succ}^2) \neq f(\overrightarrow{\succ}^3)$, where $\overrightarrow{\succ}^3$ is their union, then f is not an MLE
- Proof: $\Pr[\overrightarrow{\succ}^3 \mid \succ^*] = \Pr[\overrightarrow{\succ}^1 \mid \succ^*] \cdot \Pr[\overrightarrow{\succ}^2 \mid \succ^*]$ ■
- Lemma: Any pairwise comparison graph whose weights are even-valued can be realized via votes
- Proof: To increase the weight on the edge (a, b), add the votes $a \succ b \succ x_1 \succ \cdots \succ x_{m-2}$ and $x_{m-2} \succ \cdots \succ x_1 \succ a \succ b$ ■

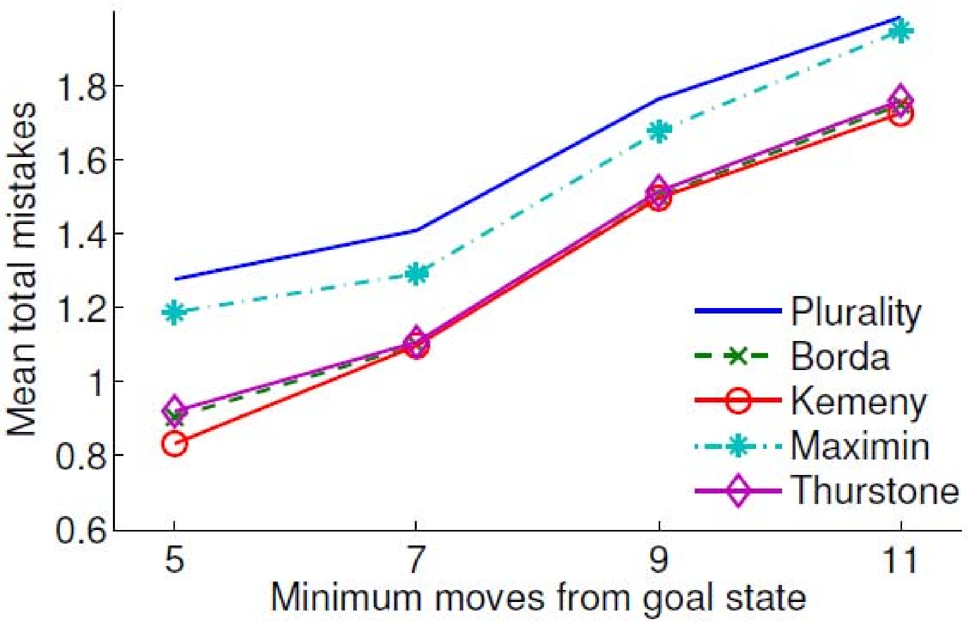
Maximin is not an MLEW

- Theorem [Conitzer and Sandholm 2005]: Maximin is not an MLE
- Proof:



Some experiments





[Mao, P, Chen 2013]