CMU 15-896

Social networks 1: Coordination Games

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Background

- Spread of ideas and new behaviors through a population
- Examples:
  - Religious beliefs and political movements
  - Adoption of technological innovations
  - Success of new product
- Process starts with early adopters and spreads through the social network

Networked coordination games

- Simple model for the diffusion of ideas and innovations
- Social network is undirected graph $G = (V, E)$
- Choice between old behavior A and new behavior B
- Parametrized by $q \in (0,1)$

Networked coordination games

- Rewards for u and v when $(u, v) \in E$:
  - If both choose A, they receive q
  - If both choose B, they receive $1 - q$
  - Otherwise both receive 0
- Overall payoff to v = sum of payoffs
- Denote d_v = degree of v, d_v^X = #neighbors playing X
- Payoff to v from choosing A is qd_v^A; reward from choosing B is $(1-q)d_v^B$
- v adopts B if $d_v^B \geq qd_v \Rightarrow q$ is a threshold

Cascading behavior

- Each node simultaneously updates its behavior in discrete time steps $t = 1,2, \dots$
- Nodes in S initially adopt B
- $h_q(S) =$ set of nodes adopting B after one round
- $h_q^k(S) =$ after k rounds of updates
- Question: When does a small set of nodes convert the entire population?

Contagion threshold

- V is countably infinite and each d_v is finite
- v is converted by S if $\exists k$ s.t. $v \in h_q^k(S)$
- S is contagious if every node is converted
- It is easier to be contagious when q is small
- Contagion threshold of $G = \max q$ s.t. $\exists$finite contagious set

Example

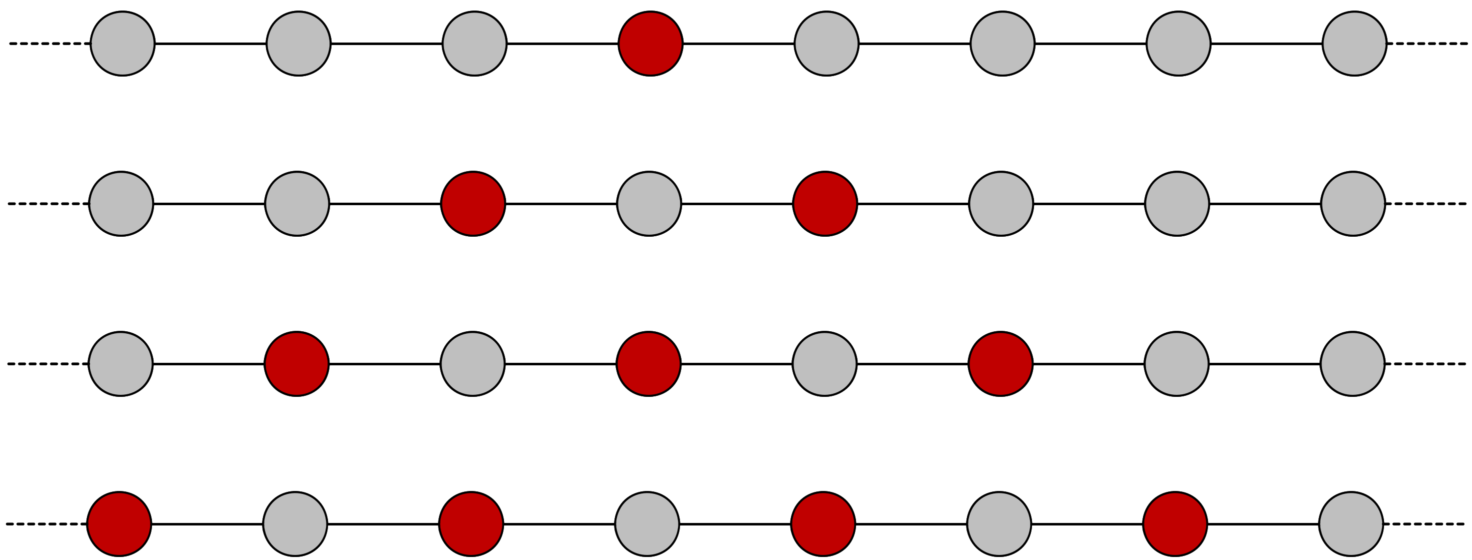
$$q = \frac{1}{2}$$

$t = 1$

$t = 2$

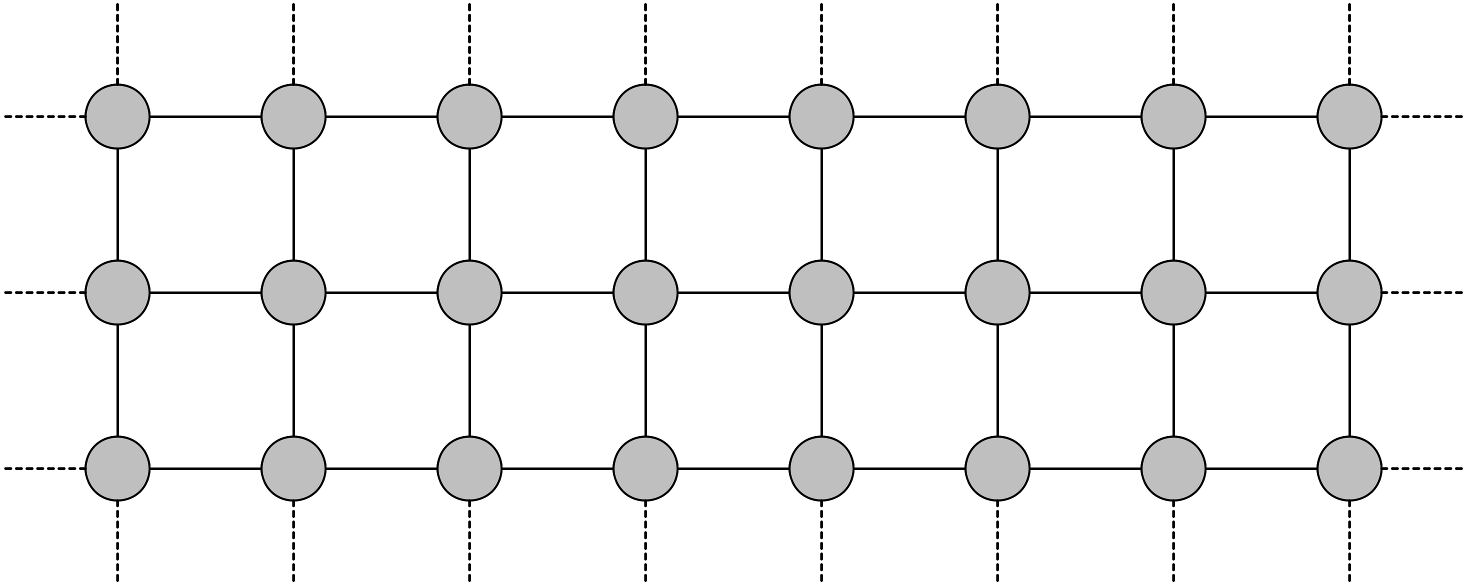
$t = 3$

$t = 4$



Poll 1: What is the contagion threshold of G?

Example



Poll 2: What is the contagion threshold of G?

Progressive processes

- Nonprogressive process: Nodes can switch from A to B or B to A
- Progressive process: Nodes can only switch from A to B
- As before, a node v switches to B if a q fraction of its neighbors $N(v)$ follow B
- $\bar{h}_q(S) =$ set of nodes adopting B in progressive process; define $\bar{h}_q^k(S)$ as before

Progressive processes

- With progressive processes intuitively the contagion threshold should be at least as high
- Theorem [Morris, 2000]: For any graph G, $\exists$finite contagious set wrt $h_q \Leftrightarrow \exists$finite contagious set wrt $\bar{h}_q$
- I.e., the contagion threshold is identical under both models

Proof of theorem

- Lemma: $\bar{h}_q^k(X) = h_q\left(\bar{h}_q^{k-1}(X)\right) \cup X$
- Proof:
  - $\bar{h}_q^k(X) = (\bar{h}_q^k(X) \setminus \bar{h}_q^{k-1}(X)) \cup (\bar{h}_q^{k-1} \setminus X) \cup X$
  - $\bar{h}_q^k(X) \setminus \bar{h}_q^{k-1}(X) = h_q\left(\bar{h}_q^{k-1}(X)\right) \setminus \bar{h}_q^{k-1}(X)$
  - For every $v \in \bar{h}_q^{k-1} \setminus X$, $v \in h_q\left(\bar{h}_q^{k-1}(X)\right)$, because v has at least as many B neighbors as when it converted
  - Clearly $X \subseteq h_q\left(\bar{h}_q^{k-1}(X)\right) \cup X$ ■

Proof of theorem

- Enough to show: given a set S that is contagious wrt $\bar{h}_q$, there is a set T that is contagious wrt h_q
- Let ℓ s.t. $S \cup N(S) \subseteq \bar{h}_q^{\ell}$; this is our T
- For $k > \ell$, $\bar{h}_q^k(S) = h_q\big(\bar{h}_q^{k-1}(S)\big) \cup S$ by the lemma
- Since $N(S) \subseteq \bar{h}_q^{k-1}(S)$, $S \subseteq h_q\big(\bar{h}_q^{k-1}(S)\big)$, and hence $\bar{h}_q^k(S) = h_q\big(\bar{h}_q^{k-1}(S)\big)$
- By induction, all $k > \ell$,

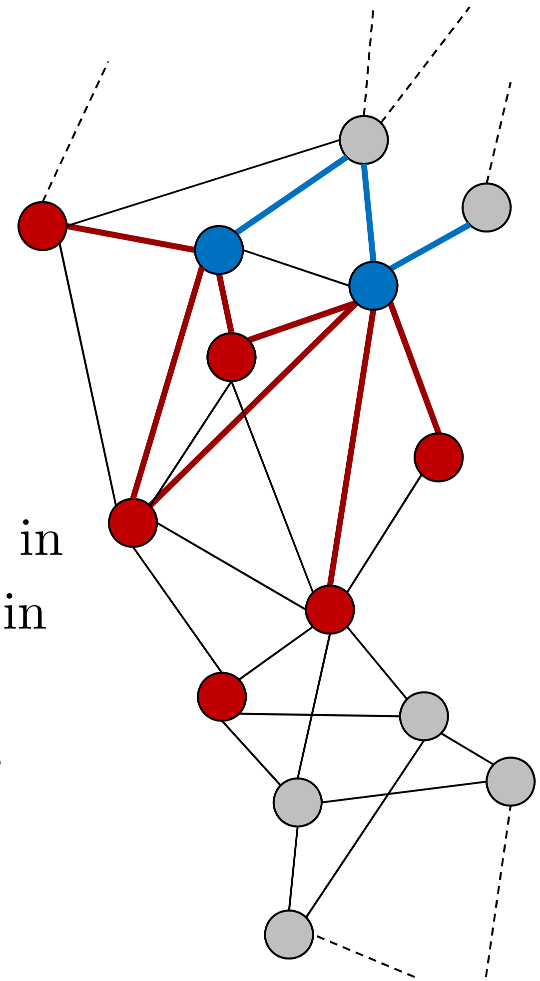
$$\bar{h}_q^k(S) = h_q^{k-\ell}\big(\bar{h}_q^{\ell}\big) = h_q^{k-\ell}(T)$$

Contagion threshold $\leq 1/2$

- Saw a graph with contagion threshold $1/2$
- Does there exist a graph with contagion threshold $> 1/2$?
- The previous theorem allows us to focus on the progressive case
- Theorem [Morris, 2000]: For any graph G, the contagion threshold $\leq 1/2$

Proof of theorem

- Let $q > 1/2$, finite S
- Denote $S_j = \bar{h}_q^j(S)$
- $\delta(X)$ = set of edges with exactly one end in X
- If $S_{j-1} \neq S_j$ then $|\delta(S_j)| < |\delta(S_{j-1})|$
    - For each $v \in S_j \setminus S_{j-1}$, its edges into S_{j-1} are in $\delta(S_{j-1}) \setminus \delta(S_j)$, and its edges into $V \setminus S_j$ are in $\delta(S_j) \setminus \delta(S_{j-1})$
    - More of the former than the latter because v converted and $q > 1/2$
- $\delta(S)$ is finite and $\delta(S_j) \geq 0$ for all j ■



More general models

- Directed graphs to model asymmetric influence
- Redefine $N(v) = \{u \in V: (u, v) \in E\}$
- Assume progressive contagion
- Node is active if it adopts B; activated if switches from A to B

Linear threshold model

- Nonnegative weight w_{uv} for each edge $(u, v) \in E$; $w_{uv} = 0$ otherwise
- Assume $\forall v \in V, \sum_u w_{uv} \leq 1$
- Each $v \in V$ has threshold θ_v
- v becomes active if

$$\sum_{\text{active } u} w_{uv} \geq \theta_v$$

General threshold model

- Linear model assumes additive influences
  - Switch if two co-workers and three family members switch?
- v has a monotonic function $g_v(\cdot)$ defined on subsets $X \subseteq N(v)$
- v becomes activated if the activated subset $X \subseteq N(v)$ satisfies $g_v(X) \geq \theta_v$

The cascade model

- When $\exists(u, v) \in E$ s.t. u is active and v is not, u has one chance to activate v
- v has an incremental function $p_v(u, X) =$ probability that u activates v when X have tried and failed
- Special cases:
  - Diminishing returns: $p_v(u, X) \geq p_v(u, Y)$ when $X \subseteq Y$
  - Independent cascade: $p_v(u, X) = p_{uv}$