CMU 15-896

Mechanism design 1: Without money

Teacher:
Ariel Procaccia

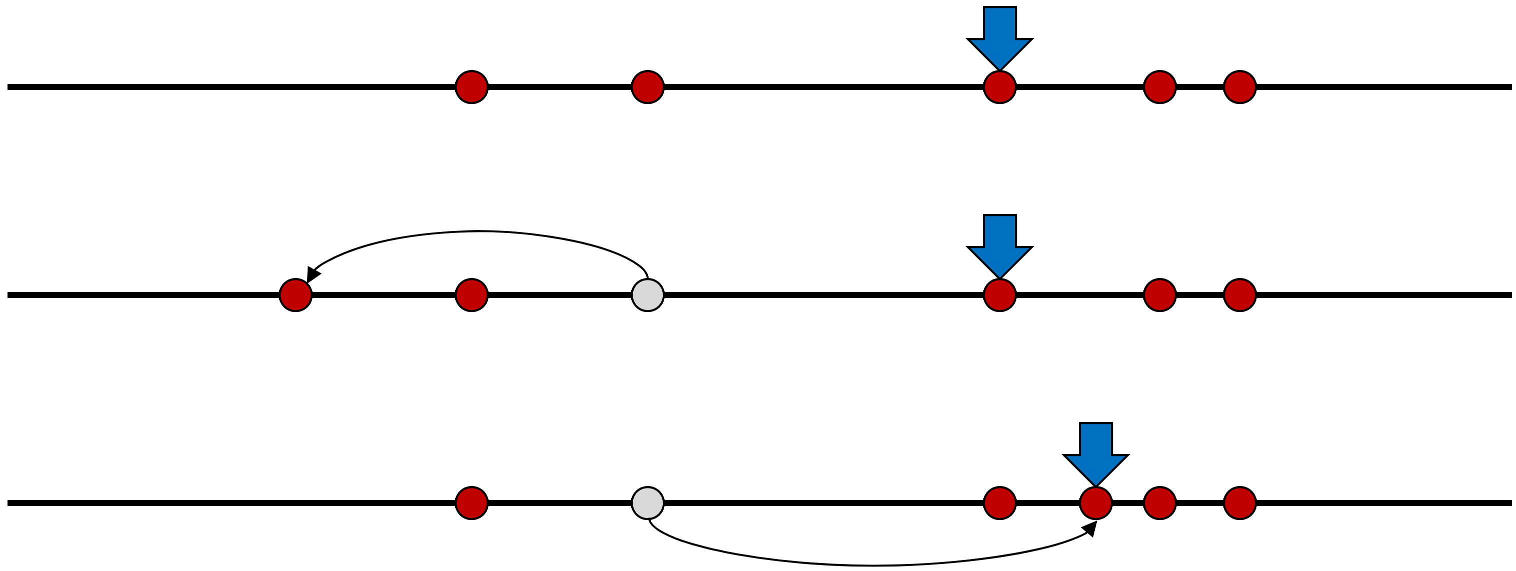
Approximate MD wo money

- We saw in kidney exchange that the optimal solution may not be strategyproof
- Approximation can be a way to quantify how much we sacrifice by insisting on strategyproofness (Example: Mix and Match)

Facility location

- Each player $i \in N$ has a location $x_i \in \mathbb{R}$
- Given $\boldsymbol{x} = (x_1, \ldots, x_n)$, choose a facility location $f(\boldsymbol{x}) = y \in \mathbb{R}$
- $\text{cost}(y, x_i) = |y - x_i|$
- Two objective functions
  - Social cost: $\text{sc}(\boldsymbol{x}) = \sum_i |y - x_i|$
  - Maximum cost: $\text{mc}(\boldsymbol{x}) = \max_i |y - x_i|$
- Social cost: the median is optimal and SP

The median is SP



MC + DET

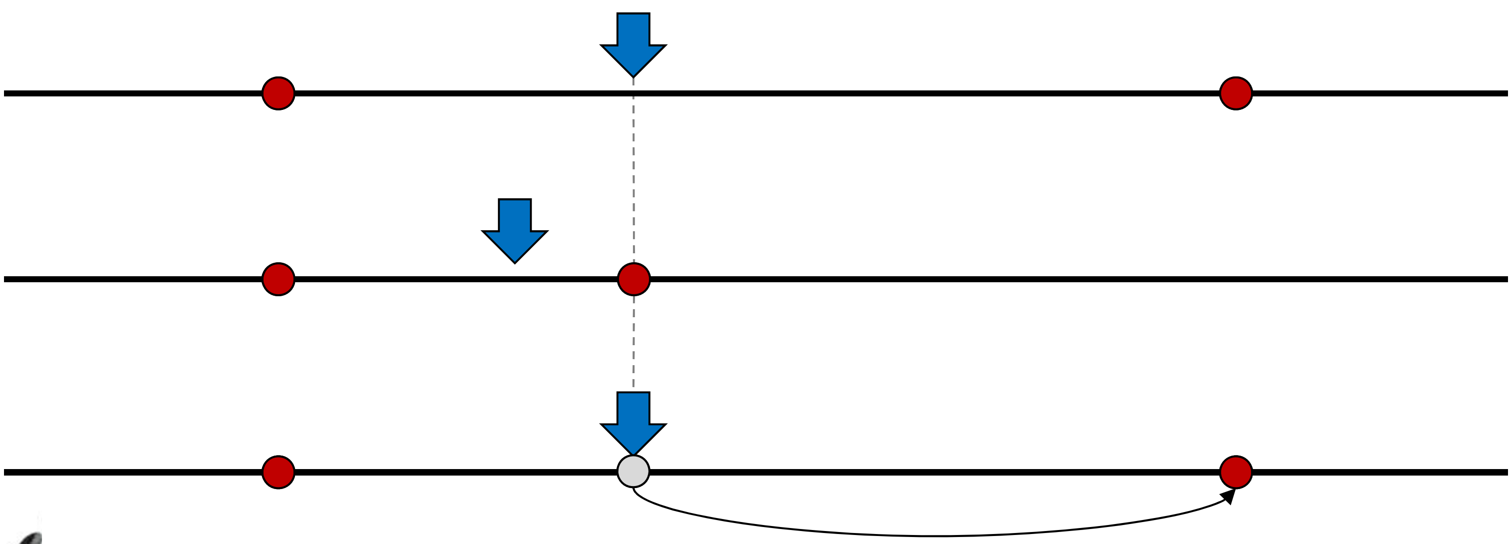
- What about maximum cost as the objective?

Poll 1: What is the approximation ratio of the median to the max cost?

1. $\in [1,2)$
2. $\in [2,3)$
3. $\in [3,4)$
4. $\in [4,\infty)$

MC + DET

- Theorem [P and Tennenholtz 2009]: No deterministic SP mechanism has an approximation ration < 2 to the max cost
- Proof:



MC + RAND

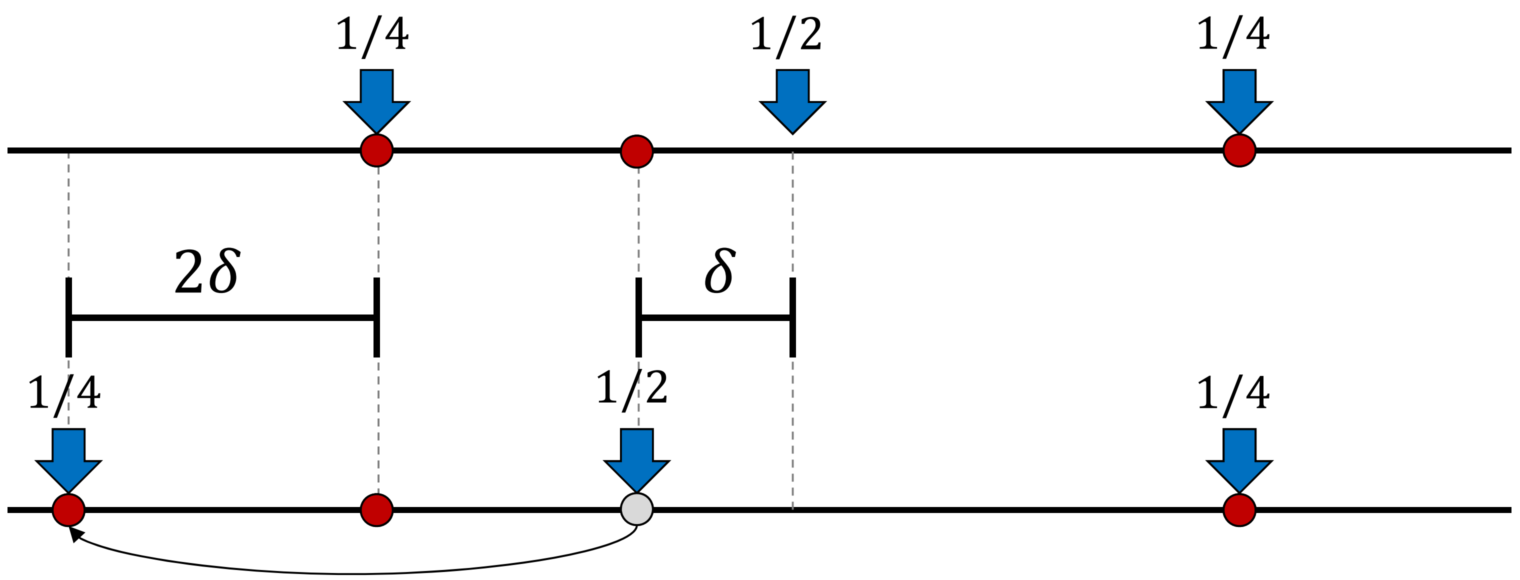
- The Left-Right-Middle (LRM) Mechanism: Choose $\min x_i$ with prob. ¼, $\max x_i$ with prob. ¼, and their average with prob. ½

Poll 2: What is the approximation ratio of the LRM Mechanism to the max cost?

1. 5/4
2. 3/2
3. 7/4
4. 2

MC + RAND

- Theorem [P and Tennenholtz 2009]: LRM is SP
- Proof:



MC + RAND

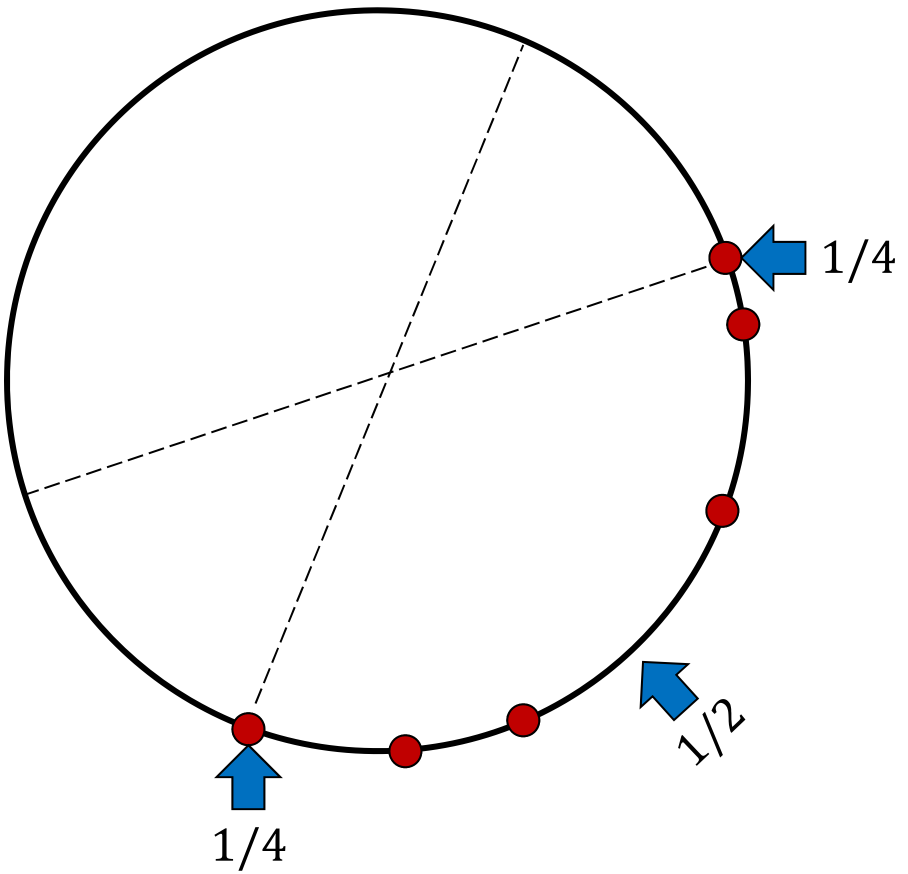
- Theorem [P and Tennenholtz 2009]: No randomized SP mechanism has an approximation ratio $< 3/2$
- Proof:
  - $x_1 = 0, x_2 = 1$, $f(\boldsymbol{x}) = P$
  - $\text{cost}(P, x_1) + \text{cost}(P, x_2) \geq 1$; wlog $\text{cost}(P, x_2) \geq 1/2$
  - $x_1 = 0, x_2' = 2$
  - By SP, the expected distance from x_2 is at least ½
  - Expected max cost at least $3/2$, because for every $y \in \mathbb{R}$, the expected cost is $|y - 1| + 1$ ■

From lines to circles

- Continuous circle
- $d(\cdot)$ is the distance on the circle
- Assume that the circumference is 1
- "Applications":
  - Telecommunications network with ring topology
  - Scheduling a daily task

MC + RAND + CIRCLE

- Semicircle like an interval on a line
- If all agents are on one semicircle, can apply LRM
- Problematic otherwise



MC + RAND + CIRCLE

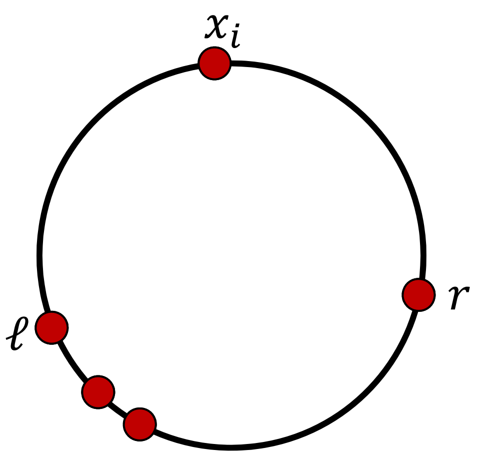
- Random Point (RP) Mechanism: Choose a random point on the circle
- Obviously horrible if players are close together
- Gives a 7/4 approx if the players cannot be placed on one semicircle
  - Worst case: many agents uniformly distributed over slightly more than a semicircle
  - If the mechanism chooses a point outside the semicircle (prob. 1/2), exp. max cost is roughly 1/2
  - If the mechanism chooses a point inside the semicircle (prob. 1/2), exp. max cost is roughly 3/8

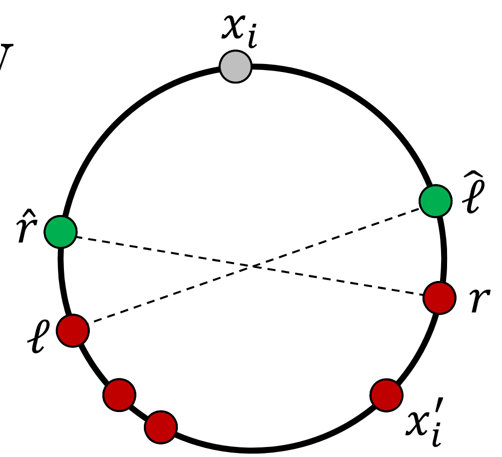
MC + RAND + CIRCLE

- Hybrid Mechanism 1: Use LRM if players are on one semicircle, RP if not
- Gives a 7/4 approx
- Surprisingly, Hybrid Mechanism 1 is also SP!

Hybrid Mechanism 1 is SP

- Deviation where RP or LRM is used before and after is not beneficial
- LRM to RP: expected cost of i is at most $1/4$ before, exactly $1/4$ after; focus on RP to LRM
- ℓ and r are extreme locations in new profile, $\hat{\ell}$ and $\hat{r}$ their antipodal points
- Because agents were not on one semicircle in $\boldsymbol{x}$, $x_i \in (\hat{\ell}, \hat{r})$

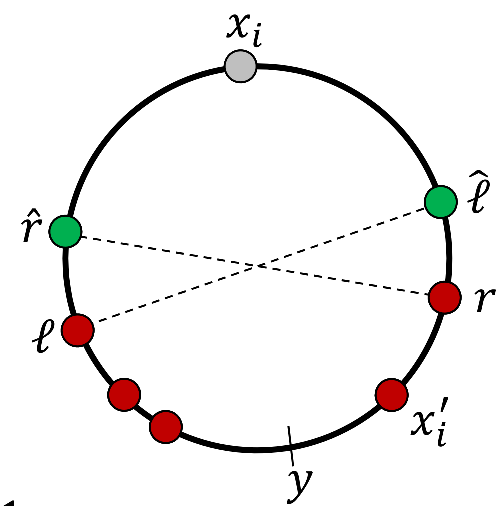




Hybrid Mechanism 1 is SP

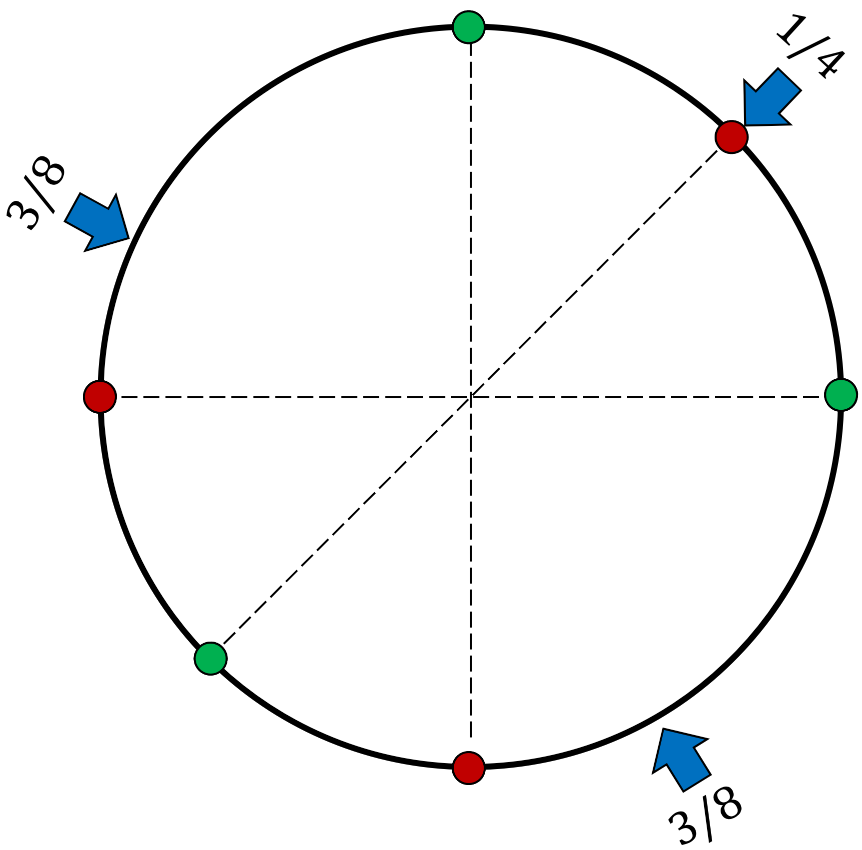
- y = center of (ℓ, r)
- $d(x_i, y) \geq 1/4$, because $d(\hat{\ell}, y) \geq 1/4$, $d(\hat{r}, y) \geq 1/4$, and $x_i \in (\hat{\ell}, \hat{r})$
- Hence,

$$\begin{aligned}\mathrm{cost}(\mathrm{lrm}(\boldsymbol{x}'), x_i) &= \frac{1}{4} d(x_i, \ell) + \frac{1}{4} d(x_i, r) + \frac{1}{2} d(x_i, y) \\ &\geq \frac{1}{4}\big(d(x_i, \ell) + d(x_i, y)\big) + \frac{1}{2} \cdot \frac{1}{4} \\ &\geq \frac{1}{4} = \mathrm{cost}(\mathrm{rp}(\boldsymbol{x}), x_i) \quad \blacksquare\end{aligned}$$



MC + RAND + CIRCLE

- Goal: improve the approx ratio of Hybrid 1?
- Random Midpoint (RM) Mechanism: choose midpoint of arc between two antipodal points with prob. proportional to length

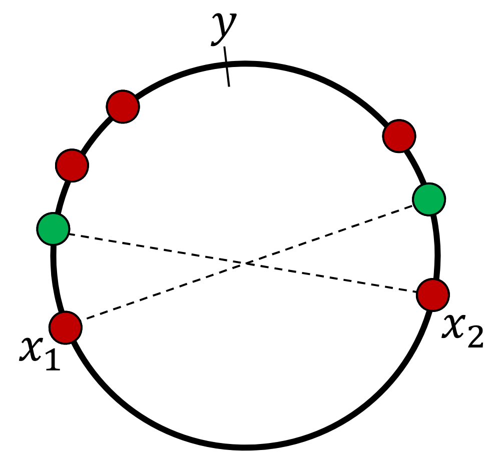


MC + RAND + CIRCLE

Poll 3: The worst example you can think of for RM gives a ratio of what to the max cost?

1. ~ 3/2
2. ~ 7/4
3. ~ 2
4. ~ 3

MC + RAND + CIRCLE



- Lemma: When the players are not on a semicircle, RM gives a $3/2$ approx
- Proof:
  - $\alpha =$ length of the longest arc between two adjacent players, w.l.o.g. x_1 and x_2
  - $\alpha \leq 1/2$ because otherwise players are on one semicircle
  - Opt y at center of $\hat{x}_1$ and $\hat{x}_2$, so $\mathrm{OPT} = (1-\alpha)/2$
  - RM selects y with probability α, and a solution with cost at most $1/2$ with prob. $1-\alpha$
  - $\dfrac{\alpha\frac{1-\alpha}{2}+\frac{1-\alpha}{2}}{\frac{1-\alpha}{2}} = 1+\alpha \leq \frac{3}{2}$ $\blacksquare$

MC + RAND + CIRCLE

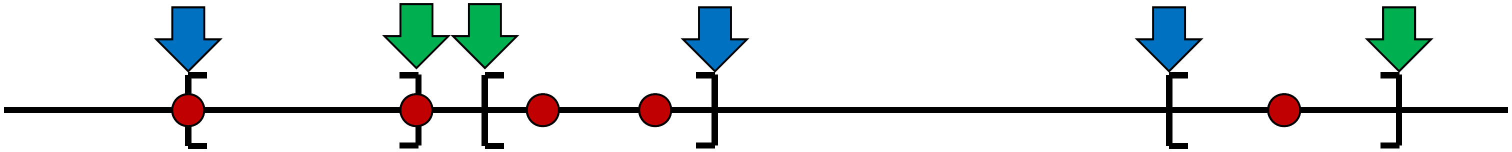
- Hybrid Mechanism 2: Use LRM if players are on one semicircle, RM if not
- Theorem [Alon et al., 2010]: Hybrid Mechanism 2 is SP and gives a 3/2 approx to the max cost
- The proof of SP is a rather tedious case analysis... but the fact that it's SP is quite amazing!

MC + RAND + $k > 1$

- Let's go back to the line, but now there are k facilities
- For $\boldsymbol{y} = (y_1, \dots, y_k)$,
$$\mathrm{cost}(\boldsymbol{y}, x_i) = \min_j |y_j - x_i|$$
- Optimal solution for max cost: cover $\boldsymbol{x}$ with k intervals of length L in a way that minimizes L; place the kth facility in the center of the kth interval

MC + RAND + $k > 1$

- Equal Cost (EC) Mechanism:
  - Cover $\boldsymbol{x}$ with k intervals as above
  - With prob. 1/2, choose the leftmost (resp., righmost) point of every odd interval, and the rightmost (resp., leftmost) point of every even interval



- Theorem [Fotakis and Tzamos 2013]: EC is an SP 2-approximation mechanism for the max cost

$k > 1$ OVERVIEW

MAX COST

	$k = 1$	$k = 2$	$2 < k < n - 1$	$k = n - 1$
Deterministic	2 [16]	2 [16]	∞ [6]	∞ [6]
Randomized	1.5 [16]	[1.5, 5/3] [16]	[1.5, **2**] [**here**]	1.5 [4]

SOCIAL COST

	$k = 1$	$k = 2$	$2 < k < n - 1$	$k = n - 1$
Deterministic	1 [13]	$n - 2$ [6], [16]	∞ [6]	∞ [6]
Randomized	1 [13]	[1.045, 4] [10], [9]	[1.045, **n**] [**here**]	[1.045, **2**] [**here**]

Fig. 1. Summary of known results on the approximability of k-Facility Location on the line (with linear cost functions) for the objectives of MAX COST and SOCIAL COST. In each cell, we have either the precise approximation ratio (if known) or the interval determined by the best known lower and upper bounds. In cells with two references, the first is for the lower bound and the second for the upper bound. We note that the lower bound on the approximation ratio of deterministic mechanisms for $k \geq 3$ is only shown for anonymous mechanisms. The randomized upper bounds proven in this work are shown in bold and hold for any concave cost function.

[Fotakis and Tzamos 2013]