Cooperative Games
The Bankruptcy Problem

Yair Zick

Based on:
Aumann & Maschler, "Game theoretic analysis of a bankruptcy problem from the Talmud", 1985

The Bankruptcy Problem

- In Judaism, a marriage is consolidated by a written contract (a *Kethubah*) between the man and the wife.
- Among other things, it stipulates the amount of money the man will leave the wife in the case of his untimely death.
- Traditionally, men are allowed to marry more than one woman; they may promise them a total amount that is more than what they have…
- A tricky legal issue (to say the least)

The Bankruptcy Problem

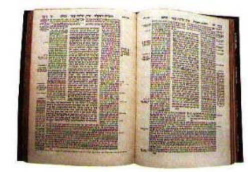
c_1

c_2

c_3

Problem: $c_1 + c_2 + c_3 > E$

The Talmudic Solution



Babylonian Talmud (300-500 CE); Tractate Ketuboth, Folio 93a (paraphrased a bit)

“If a man who was married to three wives died, and the kethubah of one wife was maneh (=100 zuz), of the second wife 200 zuz, and of the third wife 300 zuz, and the estate was worth only maneh, the sum is divided equally.

If the estate was worth 200 zuz, the claimant of the 100 zuz receives 50 and the claimants (respectively) of the 200 zuz and the 300 zuz receive 3 gold denarii (=75 zuz) each.

If the estate was worth 300 zuz, the claimant of the 100 zuz receives 50 zuz, and the claimant of the 200 zuz receives 100 zuz; the claimant of the 300 zuz receives six gold denarii (= 150 zuz).

Similarly, if three persons contributed to a joint fund and they had made a loss or a profit they share in the same manner”

The Bankruptcy Problem

$c_1 = 100$
$x_1 = 33.3$

$c_2 = 200$
$x_2 = 33.3$

$c_3 = 300$
$x_3 = 33.3$

$E = 100$

The Bankruptcy Problem

$c_1 = 100$

$x_1 = 50$

$c_2 = 200$

$x_2 = 75$

$c_3 = 300$

$x_3 = 75$

$E = 200$

The Bankruptcy Problem

$$c_1 = 100$$
$$x_1 = 50$$

$$c_2 = 200$$
$$x_2 = 100$$

$$c_3 = 300$$
$$x_3 = 150$$

$$E = 300$$

Properties of the Solution

$E =$	$c_1 = 100$	$c_2 = 200$	$c_3 = 300$
100	33.3	33.3	33.3
200	50	75	75
300	50	100	150

- Clearly not proportional to the claims: when the estate is very small it proposes an equal split.
- Mentions that the same solution should apply when costs are shared rather than rewards:

“Similarly, if three persons contributed to a joint fund and they had made a loss or a profit they share in the same manner”

Two Person Variant: Contested Garment

Babylonian Talmud; Tractate Baba Mezi’a, Folio 2a (paraphrased a bit)

“Two persons appearing before a court hold a garment.

[…] one of them says, 'it is all mine', and the other says, 'it is all mine', […] and the value of the garment shall then be divided equally between them.

If one says, 'it is all mine', and the other says, 'half of it is mine', […] the former then receives three quarters [of the value of the garment] and the latter receives one quarter.

Two Person Variant

A fair allocation:

- if p_1 and p_2 claim 100% of the estate, they get 50% each.
- if p_1 claims 100% and p_2 claims 50%, then p_2 concedes that 50% of the estate belongs to p_1: the contested 50% is then divided equally.

Two Person Variant

More generally:

- We are given $c_1 \leq c_2$ s.t. $c_1 + c_2 > E$.
- If $c_1 \geq E$ then $x_1 = x_2 = \frac{E}{2}$
- If $\mathrm{c}_1 \leq E \leq c_2$ then

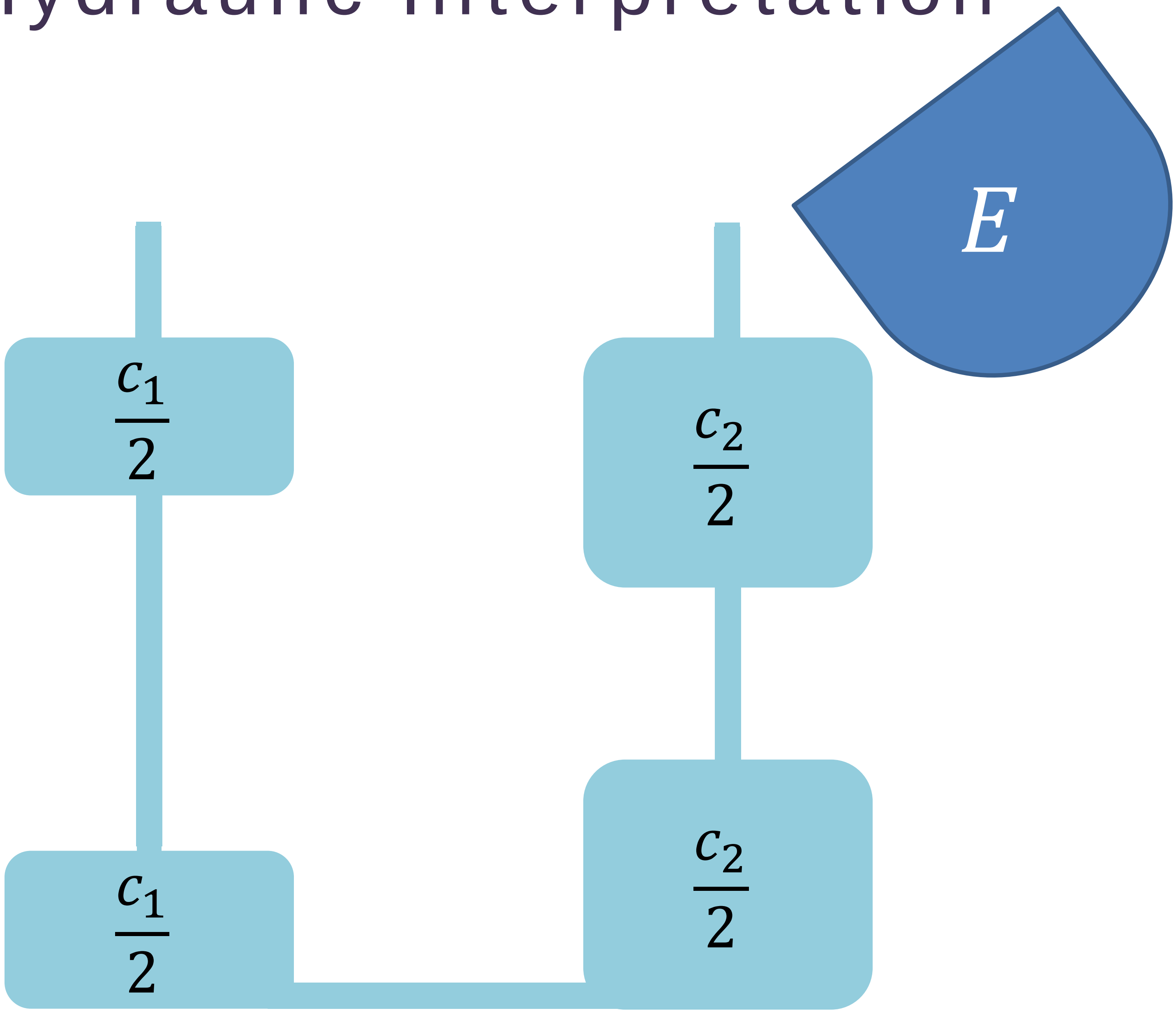
$$x_1 = \frac{c_1}{2}, x_2 = (E - c_1) + \frac{c_1}{2}$$

- Otherwise,

$$x_1 = (E - c_2) + \frac{c_1 + c_2 - E}{2};$$

$$x_2 = (E - c_1) + \frac{c_1 + c_2 - E}{2}$$

Hydraulic Interpretation



The General Case

$E =$	$c_1 = 100$	$c_2 = 200$	$c_3 = 300$
100	33.3	33.3	33.3
200	50	75	75
300	50	100	150

Observe the table:

- For every solution, and every pair of creditors, the total payoff to that pair is divided according to the 2-creditor case.
- Let us call the division rule used for the 2 person case $CG(c_1, c_2; E)$.

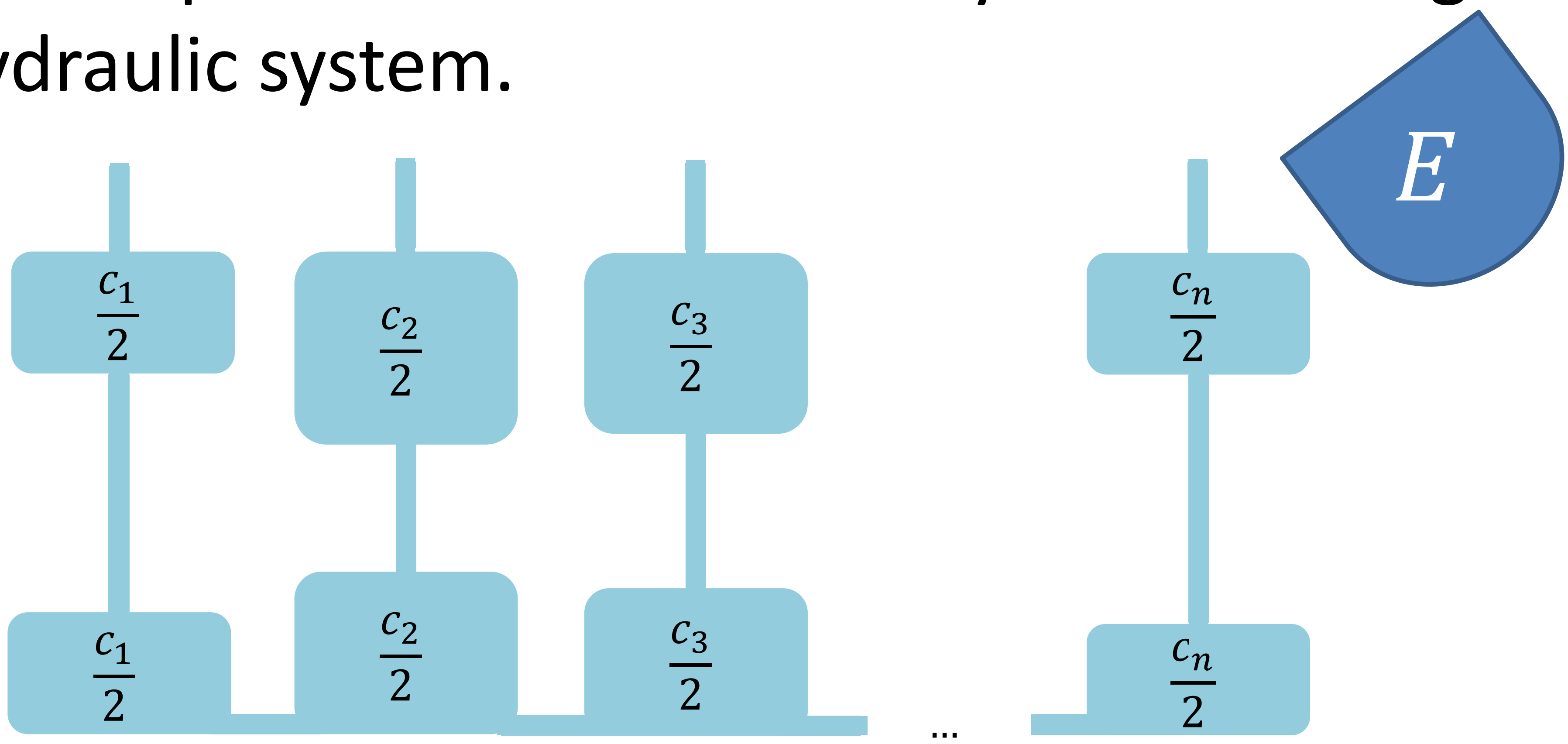
The General Case

- A *bankruptcy problem* is defined by the tuple $\langle c_1, \ldots, c_n; E \rangle$. A *solution concept* for the bankruptcy problem is a function whose input is a bankruptcy problem, and whose output is a vector in $\mathbb{R}^n$.
- We say that a solution concept φ for the bankruptcy problem is *consistent* with the Contested Garment solution if for every i, j we have

$$\varphi_i(c_1, \ldots, c_n; E) = CG(c_i, c_j; \varphi_i + \varphi_j)$$

The General Case

Theorem: there is a unique solution concept for the bankruptcy problem that is consistent with the CG problem. It is defined by the following hydraulic system.



The Nucleolus

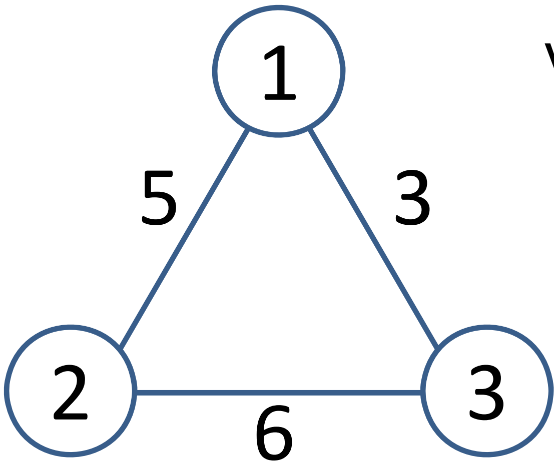
Given a cooperative game $\mathcal{G} = \langle N, v \rangle$, and a vector $\mathbf{x} \in \mathbb{R}^n$, the *excess* of a set $S \subseteq N$ with respect to $\mathbf{x}$ is $e(S, \mathbf{x}) = v(S) - x(S)$:

- Low excess: good for S
- High excess: bad for S
- The core is not empty iff the excess of every set is non-positive.
- Given an imputation $\mathbf{x}$ we write $\theta(\mathbf{x})$ to be the vector of excesses arranged from highest to lowest.

The Nucleolus

Given imputations $\mathbf{x}, \mathbf{y}$, we say that $\mathbf{x} \leq_{lex} \mathbf{y}$ if $\theta(\mathbf{x}) \leq_L \theta(\mathbf{y})$; here, $\leq_L$ denotes the lexicographic comparison of $\theta(\mathbf{x})$ and $\theta(\mathbf{y})$.

The Nucleolus



$\forall i, v(\{i\}) = 0;$ $v(\{1,2\}) = 5;$

$v(N) = 14$ $v(\{1,3\}) = 3;$

$v(\{2,3\}) = 6;$

	$\mathrm{x} = (3,4,7)$	$y = (4,8,2)$
$\{1\}$	$e(\{1\}, \mathbf{x}) = -3$	$e(\{1\}, \mathbf{y}) = -4$
$\{2\}$	$e(\{2\}, \mathbf{x}) = -4$	$e(\{2\}, \mathbf{y}) = -8$
$\{3\}$	$e(\{3\}, \mathbf{x}) = -7$	$e(\{3\}, \mathbf{y}) = -2$
$S_1 = \{1,2\}$	$e(S_1, \mathbf{x}) = -2$	$e(S_1, \mathbf{y}) = -7$
$S_2 = \{1,3\}$	$e(S_2, \mathbf{x}) = -7$	$e(S_2, \mathbf{y}) = -2$
$S_3 = \{2,3\}$	$e(S_3, \mathbf{x}) = -5$	$e(S_3, \mathbf{y}) = -4$
	$\theta(\mathbf{x}) = (-2, -3, -4 - 5, -7, -7)$	$\theta(\mathbf{y}) = (-2, -2, -4, -4, -7, -8)$

The Nucleolus

The nucleolus of a cooperative game $\mathcal{G} = \langle N, v \rangle$, denoted $Nuc(\mathcal{G})$, is an imputation that is lexicographically minimal under the order $\leq_{lex}$, i.e. for any imputation $\mathbf{x}$, we have $Nuc(\mathcal{G}) \leq_{lex} \mathbf{x}$

- The nucleolus minimizes unhappiness: any other outcome makes some set less happy.
- $Nuc(\mathcal{G})$ is unique.
- If the core is not empty, then $Nuc(\mathcal{G}) \subseteq Core(\mathcal{G})$.

The Pre-Nucleolus

The pre-nucleolus is defined in exactly the same way as the nucleolus, but we drop the assumption that $\mathbf{x}$ needs to be an imputation:

$$Nuc(\mathcal{G}) = \text{argmin}\{\theta(\mathbf{x}) \mid x(N) = v(N), \\ \forall i \in N\colon\ x_i \geq v(i)\}$$

$$PreNuc(\mathcal{G}) = \text{argmin}\{\theta(\mathbf{x}) \mid x(N) = v(N)\}$$

Consistency of the Nucleolus

Given a game $\mathcal{G} = \langle N, v \rangle$, a set $T \subseteq N$ and an imputation $\mathbf{x} \in \mathbb{R}^n$, we define the *reduced game* $\mathcal{G}|_T^{\mathbf{x}} = \langle T, v|_T^{\mathbf{x}} \rangle$, as follows:

$v|_T^{\mathbf{x}}(\emptyset) = 0$ and $v|_T^{\mathbf{x}}(T) = v(N) - x(N \setminus T)$

For all non-empty $S \subset T$:

$$v\Big|_T^{\mathbf{x}}(S) = \max_{Q \subseteq N \setminus T} \{v(S \cup Q) - x(Q)\}$$

Intuitively: players got the payoff division $\mathbf{x}$, and all players in $N \setminus T$ are happy with their share. Players in T are now trying to renegotiate their payoff division internally, and are allowed to leverage their connections in $N \setminus T$.

Consistency of the Nucleolus

A solution ϕ (i.e. a function from a game over n players to a vector in $\mathbb{R}^n$) is said to be *consistent* if for any game $\mathcal{G} = \langle N, v \rangle$, any set $T \subseteq N$

$$\phi(\mathcal{G}\left|_T^{\phi(\mathcal{G})}\right.) = \phi\left|_T\right.$$

In other words, if we take the solution and apply it to the reduced game, the payoffs remain the same (players in T would not want to renegotiate their payoffs).

Consistency of the Nucleolus

Theorem: the pre-nucleolus is consistent

Definition: a game is called 0-monotone if for any $S \subseteq N$ and any $i \in N \setminus S$,

$$v(S \cup \{i\}) \geq v(S) + v(\{i\})$$

Theorem: if a game is 0 monotone, then its nucleolus and pre-nucleolus coincide.

…So in 0 monotone games, the nucleolus is consistent.

The Nucleolus of the Bankruptcy Problem

Given a bankruptcy problem $\langle c_1, \dots, c_n; E \rangle$, we define the following cooperative game:

- $N = \{1, \dots, n\}$
- $v_{\mathbf{c};E}(S) = \max\{E - c(N \setminus S), 0\}$

The value of S is the amount of money it can claim as its own after the demands of all other members of N have been satisfied.

The Nucleolus of the Bankruptcy Problem

Theorem: $Nuc(N, v_{br})$ is the unique solution concept that is consistent with the CG problem.

Before proof: how cool is this result?

- Relates a fair payoff division rule invented ~1700 years ago with a novel game-theoretic concept.
- Probably not why it was chosen (hydraulic interpretation could be why…)

The Nucleolus of the Bankruptcy Problem

Lemma: $Nuc(N, v_{br})$ coincides with the CG solution for two persons.

Proof idea:

1. Write out the possible values of v_{br} for the two player game .
2. Characterize the nucleolus for two players.

The Nucleolus of the Bankruptcy Problem

Lemma: Let $\mathbf{x}$ be the solution to the bankruptcy problem of $\langle c_1, \ldots, c_n; E \rangle$; then for any set $S \subseteq N$, the reduced game $v_{\mathbf{c};E}|_S^{\mathbf{x}}$ coincides with the bankruptcy problem game for S where the claims are $\mathbf{c}|_S = (c_i)_{i \in S}$ and the estate is $x(S)$

In other words, for any non-empty $T \subset S$

$$\max_{Q \subseteq N \setminus S} \{v_{\mathbf{c},E}(T \cup Q) - x(Q)\} = v_{\mathbf{c}|_S, x(S)}(T)$$

The Nucleolus of the Bankruptcy Problem

Proof of Theorem: let $\mathbf{y}$ be the nucleolus of $v_{\mathbf{c},E}$ and let $S = \{i,j\}$ be any two player coalition.
Since the bankruptcy game is 0-monotone, $\mathbf{y}$ is consistent: (y_i, y_j) is the nucleolus of the reduced game $v|_S^{\mathbf{y}}$, which equals the CG game $v_{(c_i,c_j),y(S)}$ for i,j. As we have shown, the bankruptcy solution for the CG game for i,j coincides with the nucleolus (i.e., it is (y_i, y_j)). So we are done!