CMU 15-896

Social choice 4: Voting Rules as MLEs

Teacher:
Ariel Procaccia

Condorcet strikes again

- For Condorcet [1785], the purpose of voting is not merely to balance subjective opinions; it is a collective quest for the truth
- Enlightened voters try to judge which alternative best serves society
- For $m = 2$ the majority opinion will very likely be correct
- Realistic in trials by jury or the pooling of expert opinions — or in human computation!

Motivation: EteRNA

- Developed at CMU (Adrien Treuille) and Stanford
- Choose 8 RNA designs to synthesize
- Some designs are truly more stable than others
- The goal of voting is to compare the alternatives by true quality

Condorcet's noise model

- True ranking of the alternatives
- Voting pairwise on alternatives, each comparison is correct with prob. $p > 1/2$
- Results are tallied in a voting matrix

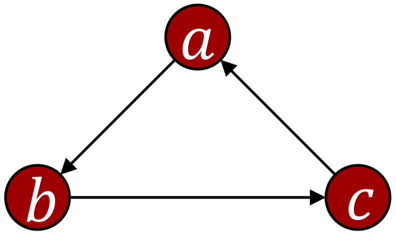
	a	b	c
a	-	8	6
b	5	-	11
c	7	2	-

What is the Borda score of alternative b?

Condorcet's 'solution'

- Condorcet's goal: find "the most probable" ranking
- Condorcet suggested: take the majority opinion for each comparison; if a cycle forms, "successively delete the comparisons that have the least plurality"
- In example, we delete $c \succ a$ to get $a \succ b \succ c$

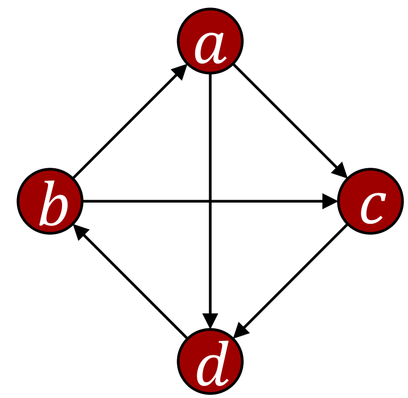
	a	b	c
a	-	8	6
b	5	-	11
c	7	2	-



Condorcet's 'solution'

- With four alternatives we get ambiguities
- In example, order of strength is $c \succ d$, $a \succ d$, $b \succ c$, $a \succ c$, $d \succ b$, $b \succ a$
- Delete $b \succ a \Rightarrow$ still cycle
- Delete $d \succ b \Rightarrow$ either a or b could be top-ranked

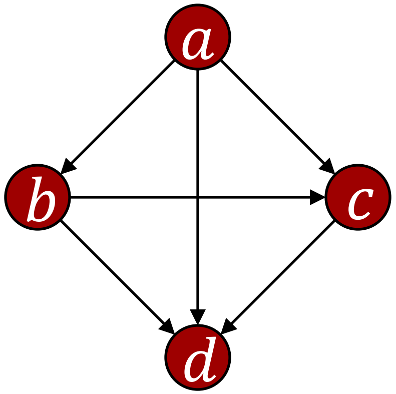
	a	b	c	d
a	-	12	15	17
b	13	-	16	11
c	10	9	-	18
d	8	14	7	-



Condorcet's 'solution'

- Did Condorcet mean we should reverse the weakest comparisons?
- Reverse $b \succ a$ and $d \succ b \Rightarrow$ we get $a \succ b \succ c \succ d$, with 89 votes
- $b \succ a \succ c \succ d$ has 90 votes (only reverse $d \succ b$)

	a	b	c	d
a	-	12	15	17
b	13	-	16	11
c	10	9	-	18
d	8	14	7	-



Exasperation?

- "The general rules for the case of any number of candidates as given by Condorcet are stated so briefly as to be hardly intelligible . . . and as no examples are given it is quite hopeless to find out what Condorcet meant" [Black 1958]
- "The obscurity and self-contradiction are without any parallel, so far as our experience of mathematical works extends ... no amount of examples can convey an adequate impression of the evils" [Todhunter 1949]

Young's solution

- Suppose true ranking is $a \succ b \succ c$; prob of observations:
$$\binom{13}{8} p^8(1-p)^5 \cdot \binom{13}{6} p^6(1-p)^7 \cdot \binom{13}{11} p^{11}(1-p)^2$$
- For $a \succ c \succ b$ prob. is:
$$\binom{13}{8} p^8(1-p)^5 \cdot \binom{13}{6} p^6(1-p)^7 \cdot \binom{13}{2} p^2(1-p)^{11}$$
- Coefficients are identical
- Exponent of p is #agreements, exponent of $1-p$ is #disagreements

	a	b	c
a	-	8	6
b	5	-	11
c	7	2	-

Young's solution

- M = matrix of votes
- $\Pr[\succ|M] = \frac{\Pr[M|\succ]\cdot\Pr[\succ]}{\Pr[M]}$
- Assume uniform prior over $\succ$, $\Pr[\succ] = \frac{1}{m!}$
- Must maximize $\Pr[M|\succ]$, do this by minimizing #disagreements with observed votes on pairs of alternatives
- This is the Kemeny rule

The Kemeny rule

- Theorem [Bartholdi, Tovey, Trick 1989]: Computing the Kemeny ranking is NP-hard
- Typically formulated as an ILP: for every $e = (a, b) \in A^2$, $x_e = 1$ iff a is ranked above b, and

$$w_e = |\{i \in N \mid a \succ_i b\}|$$

The Kemeny rule

Maximize $\sum_e x_e w_e$

Subject to

For all distinct $a, b \in A, x_{(a,b)} + x_{(b,a)} = 1$

For all distinct $a, b, c \in A, x_{(a,b)} + x_{(b,c)} + x_{(c,a)} \leq 2$

For all distinct $a, b \in A, x_{(a,b)} \in \{0,1\}$

Ten years later...

- Noise model = distribution over votes (rankings) for each true ranking
- Votes are drawn independently
- Which voting rules have a noise model for which they are MLEs of the true ranking?

Turning the question on its head

Scoring rules as MLEWs

- Theorem [Conitzer and Sandholm 2005]: Any scoring rule is an MLE
- Proof:
  - $x_1 \succ^* x_2 \succ^* \cdots \succ^* x_m$ = true ranking
  - The probability that a voter i ranks each alternative x_j in position r_{ij} is prop. to

$$\prod_{j=1}^{m} (m+1-j)^{s_{r_{ij}}}$$

Scoring rules as MLEWs

- Proof (continued):
  - $\Pr[M \mid \prec^*] \propto \prod_{i=1}^{n} \prod_{j=1}^{m} (m+1-j)^{s_{r_{ij}}}$
  - This is equal to

$$\prod_{j=1}^{m} (m+1-j)^{\sum_{i=1}^{n} s_{r_{ij}}}$$

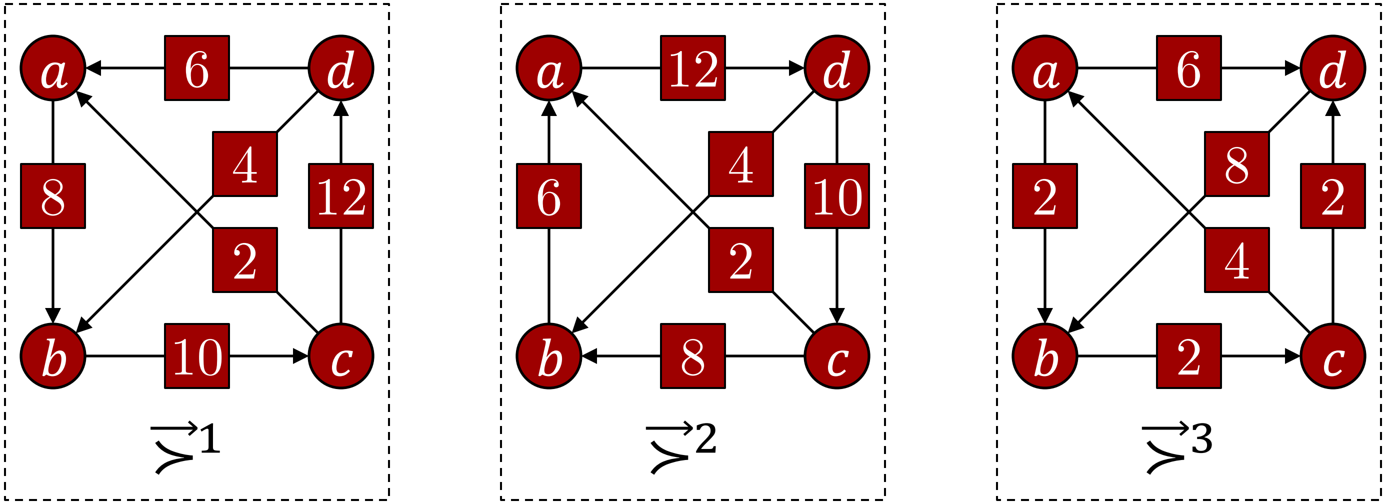
  - $m+1-j$ is positive and decreasing in j, so to maximize label alternative with kth highest score as x_k ■

Maximin is not an MLEW

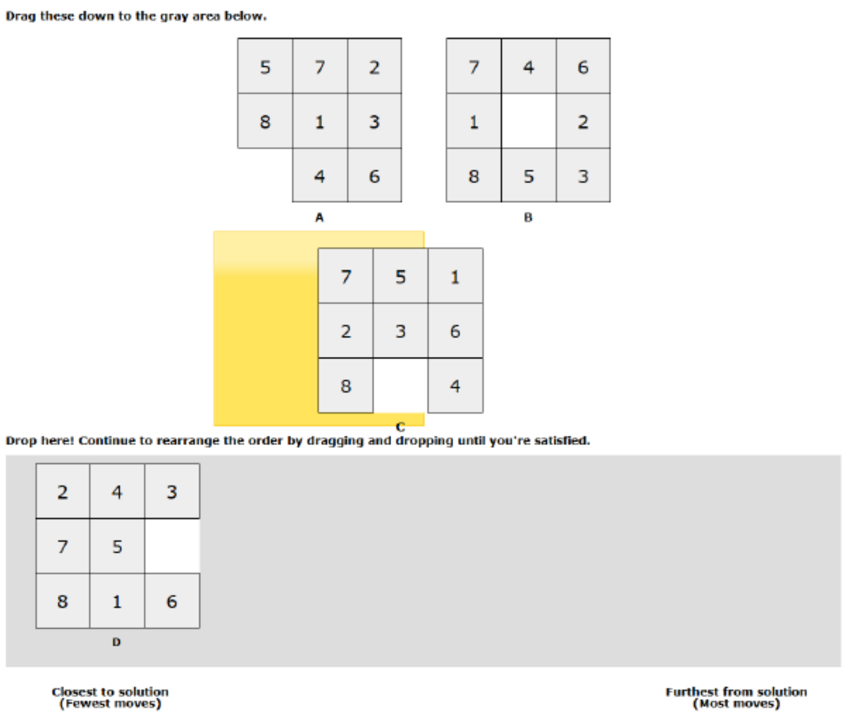
- Lemma: If there exist preference profiles $\vec{\succ}^1$ and $\vec{\succ}^2$ such that $f(\vec{\succ}^1) = f(\vec{\succ}^2) \neq f(\vec{\succ}^3)$, where $\vec{\succ}^3$ is their union, then f is not an MLE
- Proof: $\Pr[\vec{\succ}^3 \mid \succ^*] = \Pr[\vec{\succ}^1 \mid \succ^*] \cdot \Pr[\vec{\succ}^2 \mid \succ^*]$ ■
- Lemma: Any pairwise comparison graph whose weights are even-valued can be realized via votes
- Proof: To increase the weight on the edge (a, b), add the votes $a \succ b \succ x_1 \succ \cdots \succ x_{m-2}$ and $x_{m-2} \succ \cdots \succ x_1 \succ a \succ b$ ■

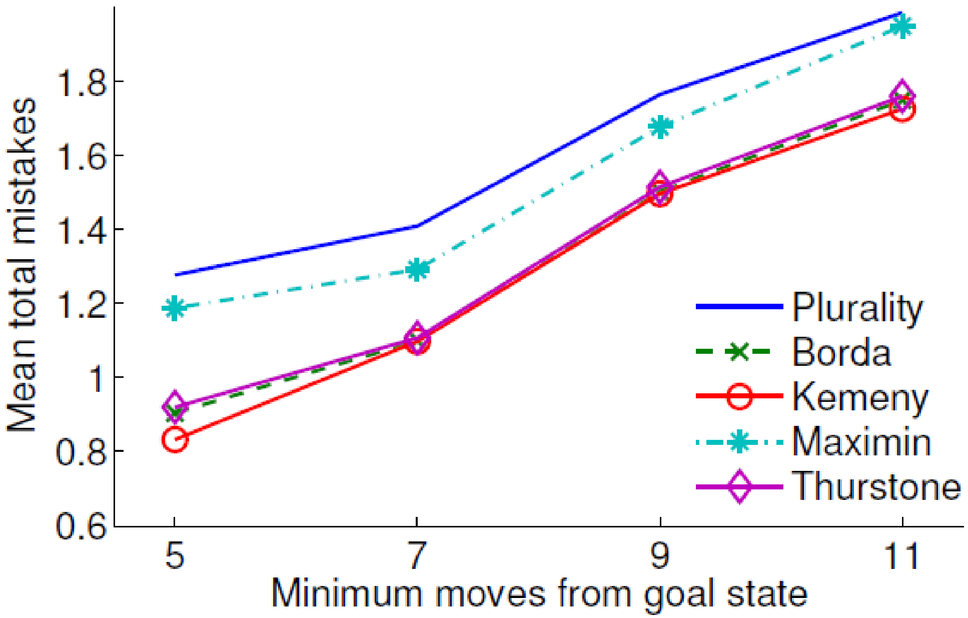
Maximin is not an MLEW

- Theorem [Conitzer and Sandholm 2005]: Maximin is not an MLE
- Proof:



Some experiments





[Mao, P, Chen 2013]