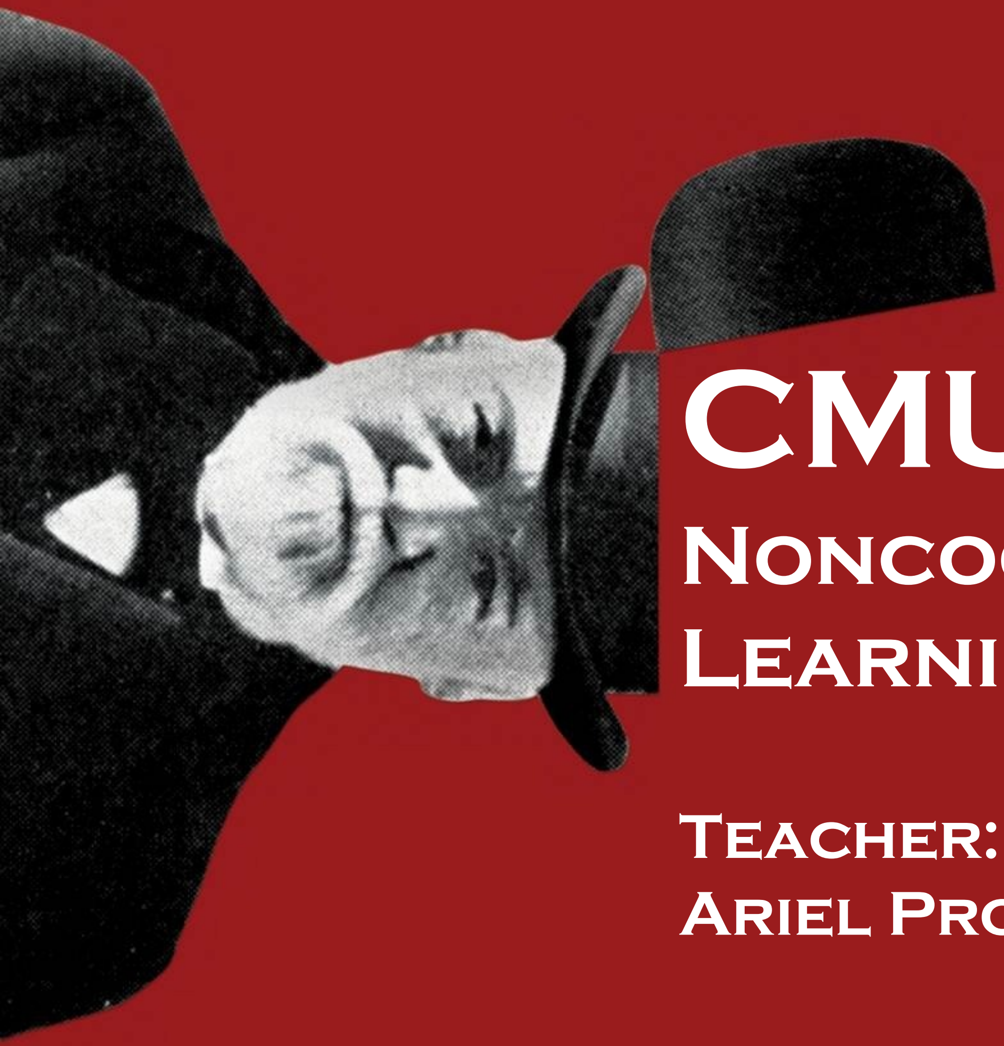


CMU 15-896

Noncooperative games 2: Learning and minimax

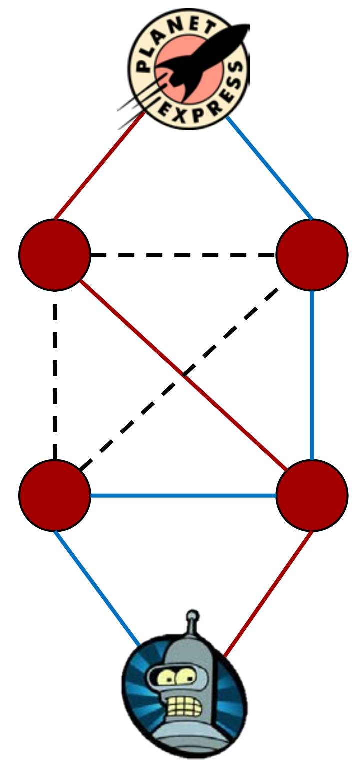
Teacher:
Ariel Procaccia

Reminder: The Minimax Theorem

- Theorem [von Neumann, 1928]: Every 2-player zero-sum game has a unique value v such that:
  - Player 1 can guarantee value at least v
  - Player 2 can guarantee loss at most v
- We will prove the theorem via no-regret learning

How to reach your spaceship

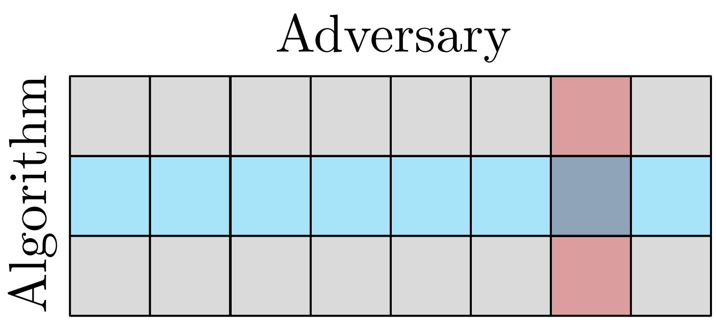
- Each morning pick one of n possible routes
- Then find out how long each route took
- Is there a strategy for picking routes that does almost as well as the best fixed route in hindsight?



53 minutes
47 minutes
…

The model

- View as a matrix (maybe infinite #columns)



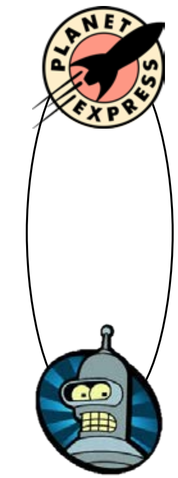
- Algorithm picks row, adversary column
- Alg pays cost of (row,column) and gets column as feedback
- Assume costs are in [0,1]

The model

- Define average regret in T time steps as (average per-day cost of alg) − (average per-day cost of best fixed row in hindsight)
- No-regret algorithm: regret $\to 0$ as $T \to \infty$
- Not competing with adaptive strategy, just the best fixed column

Example

- Algorithm 1: Alternate between U and D

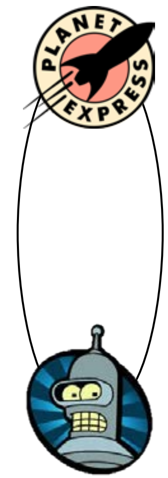


- Poll 1: What is algorithm 1's worst-case average regret?
  1. $\Theta(1/T)$
  2. $\Theta(1)$
  3. $\Theta(T)$
  4. ∞

	Adversary	
Algorithm	1	0
	0	1

Example

- Algorithm 2: Choose action that has lower cost so far



- Poll 2: What is algorithm 2's worst-case average regret?
  1. $\Theta(1/T)$
  2. $\Theta(1/\sqrt{T})$
  3. $\Theta(1/\log T)$
  4. $\Theta(1)$

	Adversary	
Algorithm	1	0
	0	1

What can we say more generally about deterministic algorithms?

Using expert advice

- Want to predict the stock market
- Solicit advice from n experts
  - Expert $=$ someone with an opinion

Day	Expert 1	Expert 2	Expert 3	Charlie	Truth
1	$-$	$-$	$+$	$+$	$+$
2	$+$	$-$	$+$	$-$	$-$
…	…	…	…	…	…

- Can we do as well as best in hindsight?

Simpler question

- One of the n experts never makes a mistake
- We want to find out which one
- Algorithm 3: Take majority vote over experts that have been correct so far
- Poll 3: What is algorithm 3's worst-case number of mistakes?
  1. $\Theta(1)$
  2. $\Theta(\log n)$
  3. $\Theta(n)$
  4. ∞

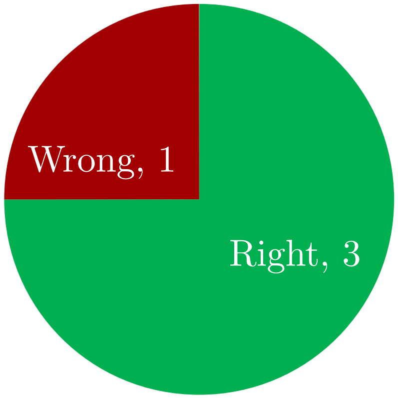
What if no expert is perfect?

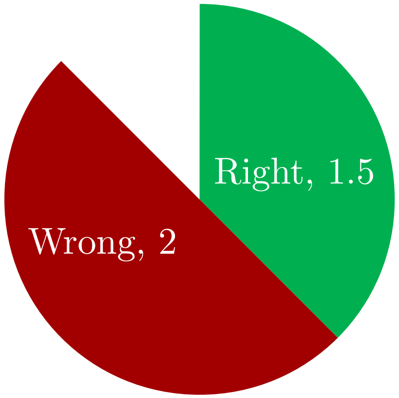
- Idea: Run algorithm 3 until all experts are crossed off, then repeat
- Makes at most $\log n$ mistakes per mistake of the best expert
- But this is wasteful: we keep forgetting what we've learned

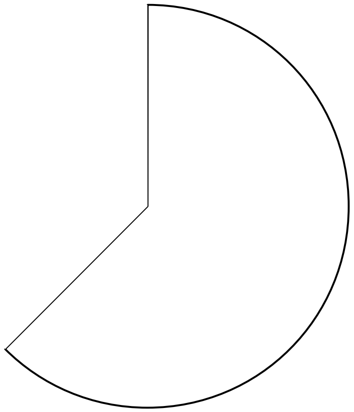
Weighted Majority

- Intuition: Making a mistake doesn't disqualify an expert, just lowers its weight
- Weighted Majority Algorithm:
  - Start with all experts having weight 1
  - Predict based on weighted majority vote
  - Penalize mistakes by cutting weight in half

	Expert 1	Expert 2	Expert 3	Charlie	Alg	Truth
Weight 1	1	1	1	1		
Prediction 1	−	+	+	+	+	+
Weight 2	0.5	1	1	1		
Prediction 2	+	+	−	−	−	+
Weight 3	0.5	1	0.5	0.5		







Weighted Majority: Analysis

- M = #mistakes we've made so far
- m = #mistakes of best expert so far
- W = total weight (starts at n)
- For each mistake, W drops by at least 25%
  $\Rightarrow$ after M mistakes: $W \leq n(3/4)^M$
- Weight of best expert is $(1/2)^m$

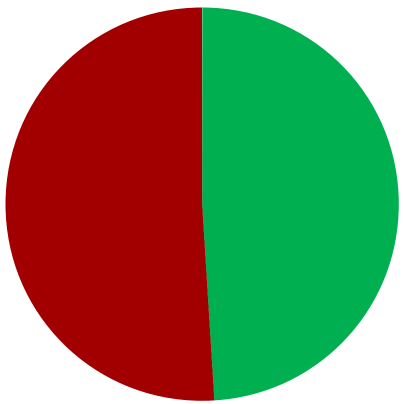
$$\left(\frac{1}{2}\right)^m \leq n\left(\frac{3}{4}\right)^M \Rightarrow \left(\frac{4}{3}\right)^M \leq n2^m \Rightarrow M \leq 2.5(m + \log n)$$

Randomized Weighted Majority

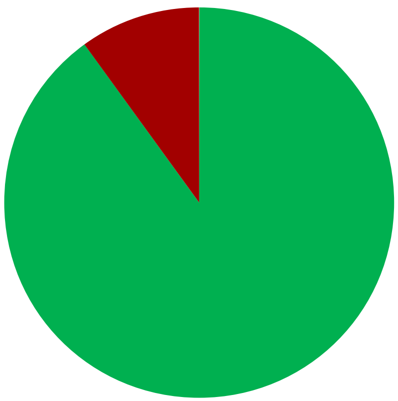
- Randomized Weighted Majority Algorithm:
  - Start with all experts having weight 1
  - Predict proportionally to weights: the total weight of $+$ is w_+ and the total weight of $-$ is w_-, predict $+$ with probability $\frac{w_+}{w_+ + w_-}$ and $-$ with probability $\frac{w_-}{w_+ + w_-}$
  - Penalize mistakes by removing ϵ fraction of weight

Randomized Weighted Majority

Idea: smooth out the worst case



The worst-case is ~50-50: now we have a 50% chance of getting it right



What about 90-10? We're very likely to agree with the majority

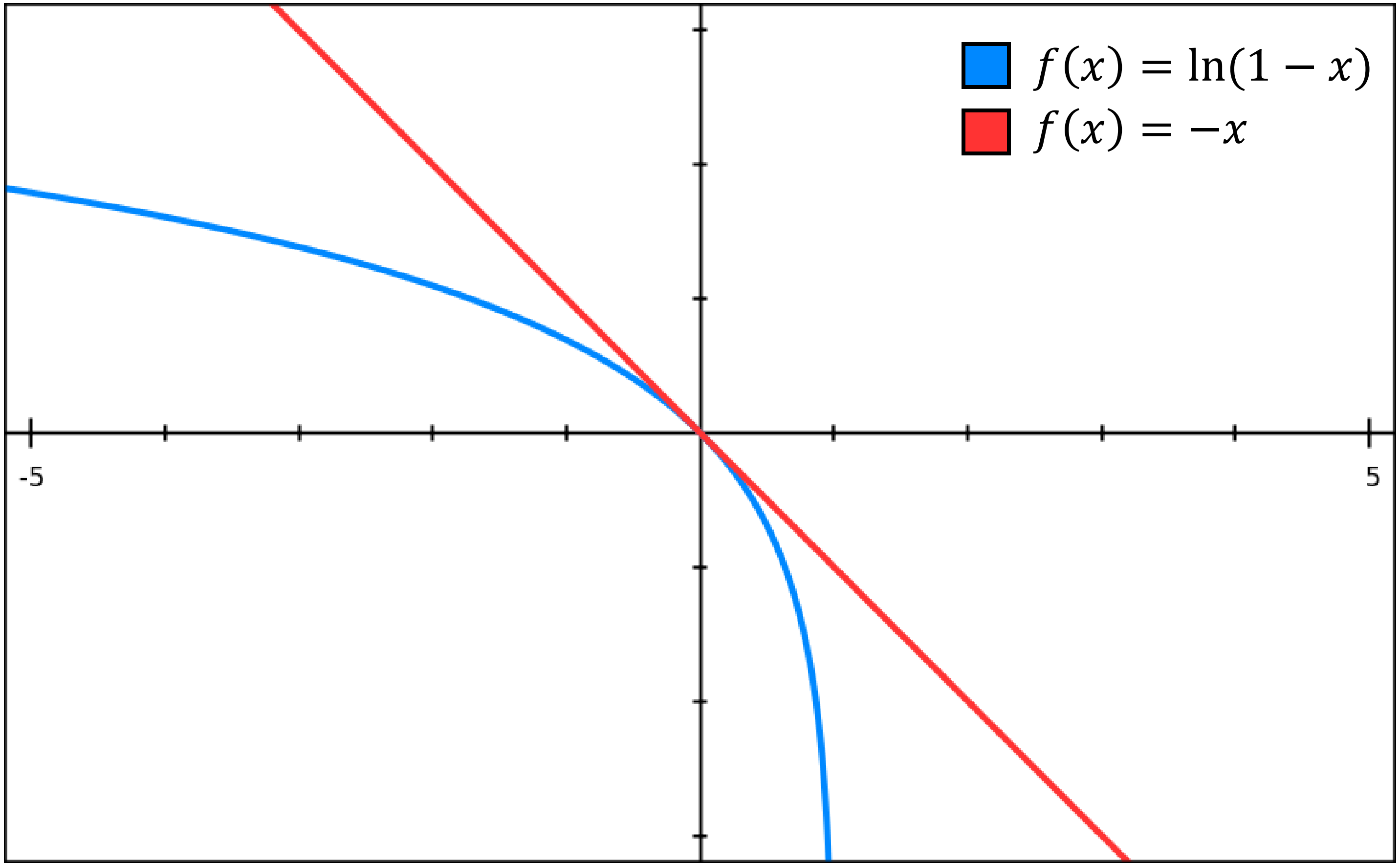
Analysis

- At time t we have a fraction F_t of weight on experts that made a mistake
- Prob. F_t of making a mistake, remove ϵF_t fraction of total weight
- $W_{final} = n \prod_t (1 - \epsilon F_t)$
- $\ln W_{final} = \ln n + \sum_t \ln(1 - \epsilon F_t)$

$$\leq \ln n - \epsilon \sum_t F_t = \ln n - \epsilon M$$

$\ln(1 - x) \leq -x$
(next slide)

Analysis



Analysis

- Weight of best expert is $W_{best} = (1-\epsilon)^m$
- $\ln n - \epsilon M \geq \ln W_{final} \geq \ln W_{best} = m \ln(1-\epsilon)$
- By setting $\epsilon = \sqrt{\frac{\log n}{m}}$ and solving, we get

$$M \leq m + 2\sqrt{m \log n}$$

- Since $m \leq T$, $M \leq m + 2\sqrt{T \log n}$
- Average regret is $\left(2\sqrt{T \log n}\right)/T \to 0$ ■

More generally

- Each expert is an action with cost in [0,1]
- Run Randomized Weighted Majority
  - Choose expert i with probability w_i/W
  - Update weights: $w_i \leftarrow w_i(1 - c_i\epsilon)$
- Same analysis applies:
  - Our expected cost: $\sum_j c_j w_j / W$
  - Fraction of weight removed: $\epsilon \sum_j c_j w_j / W$
  - So, fraction removed $= \epsilon \cdot$ (our cost)

Proof of the minimax thm

- Suppose for contradiction that zero-sum game G has $V_C > V_R$ such that:
  - If column player commits first, there is a row that guarantees row player at least V_C
  - If row player commits first, there is a column that guarantees row player at most V_R
- Scale matrix so that payoffs to row player are in $[-1,0]$, and let $V_C = V_R + \delta$

Proof of the minimax thm

- Row player plays RWM, and column player responds optimally to current mixed strategy
- After T steps
  - $\text{ALG} \geq$ best row in hindsight $-2\sqrt{T \log n}$
  - Best row in hindsight $\geq T \cdot V_C$
  - $\text{ALG} \leq T \cdot V_R$
- It follows that $T \cdot V_R \geq T \cdot V_C - 2\sqrt{T \log n}$
- $\delta T \leq 2\sqrt{T \log n}$ – contradiction for large enough T ■