CMU 15-896

Fair division 3: Rent division

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Strategyproof Cake cutting

- We discussed strategyproofness (SP) in voting
- All the cake cutting algorithms we discussed are not SP: agents can gain from manipulation
  - Cut and choose: player 1 can manipulate
  - Dubins-Spanier: shout later
- Assumption: agents report full valuations
- Deterministic EF and SP algs exist in some special cases, but they are rather involved [Chen et al. 2010]

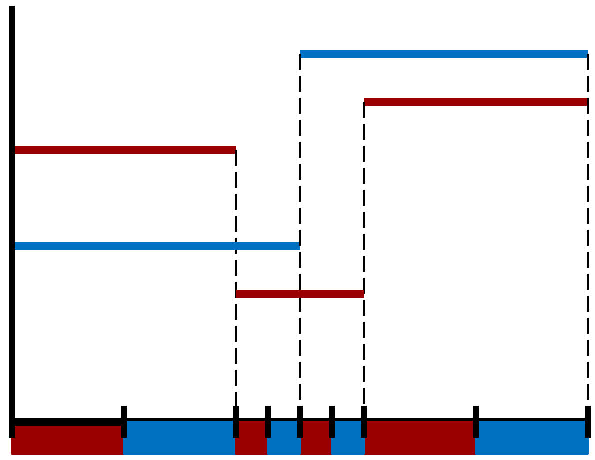
A randomized algorithm

- $X_1, \dots, X_n$ is a perfect partition if $V_i(X_j) = 1/n$ for all i, j
- Algorithm
  - Compute a perfect partition
  - Draw a random permutation π over $\{1, \dots, n\}$
  - Allocate to agent i the piece $X_{\pi(i)}$
- Theorem [Chen et al. 2010; Mossel and Tamuz 2010]: the algorithm is SP in expectation and always produces an EF allocation
- Proof: if an agent lies the algorithm may compute a different partition, but for any partition:

$$\sum_{j \in N} \frac{1}{n} V_i(X_j') = \frac{1}{n} \sum_{j \in N} V_i(X_j') = \frac{1}{n} \quad \blacksquare$$

Computing a perfect partition

- Theorem [Alon, 1986]: a perfect partition always exists, needs polynomially many cuts
- Proof is nonconstructive
- Can find perfect partitions for special valuation functions

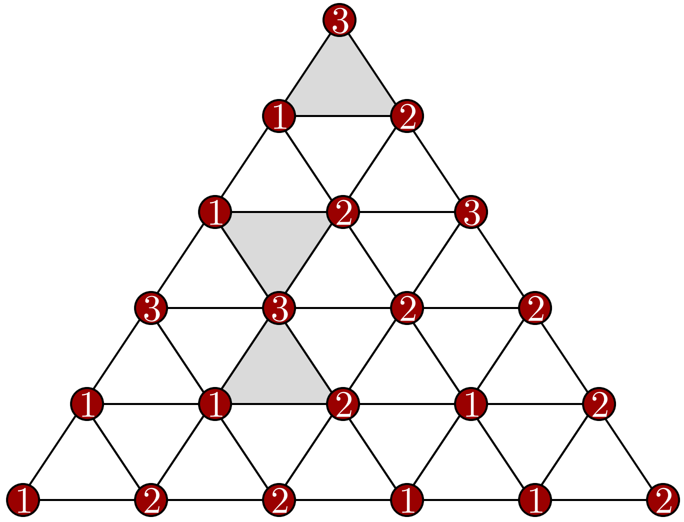


A true story

- In 2001 I moved into an apartment in Jerusalem with Naomi and Nir
- One larger bedroom, two smaller bedrooms
- Naomi and I searched for the apartment, Nir was having fun in South America
- Nir's argument: I should have the large room because I had no say in choosing apartment
    - Made sense at the time!
- How to fairly divide the rent?

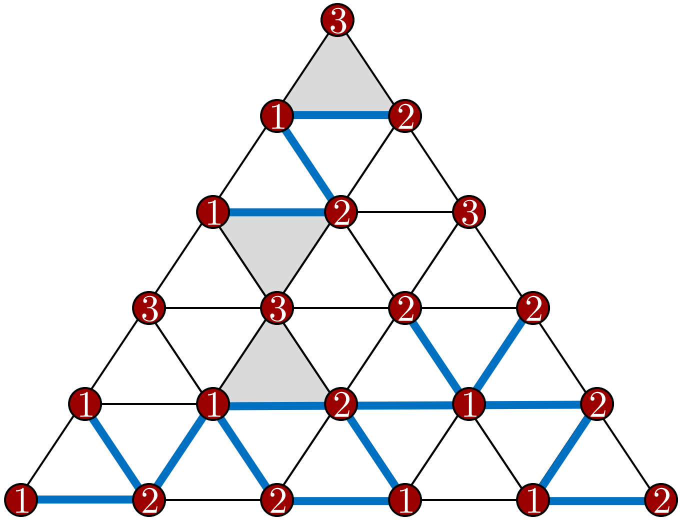
Sperner's Lemma

- Triangle T partitioned into elementary triangles
- Label vertices by {1,2,3} using Sperner labeling:
  - Main vertices are different
  - Label of vertex on an edge (i, j) of T is i or j
- Lemma: Any Sperner labeling contains at least one fully labeled elementary triangle



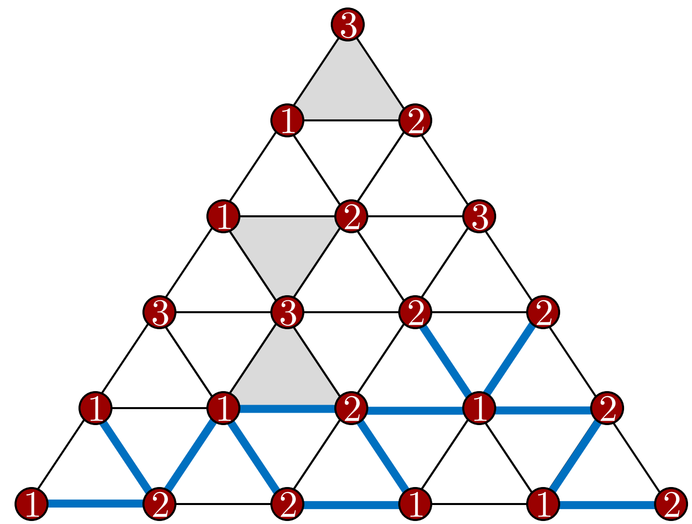
Proof of Lemma

- Doors are 12 edges
- Rooms are elementary triangles
- #doors on the boundary of T is odd
- Every room has ≤ 2 doors; one door iff the room is 123



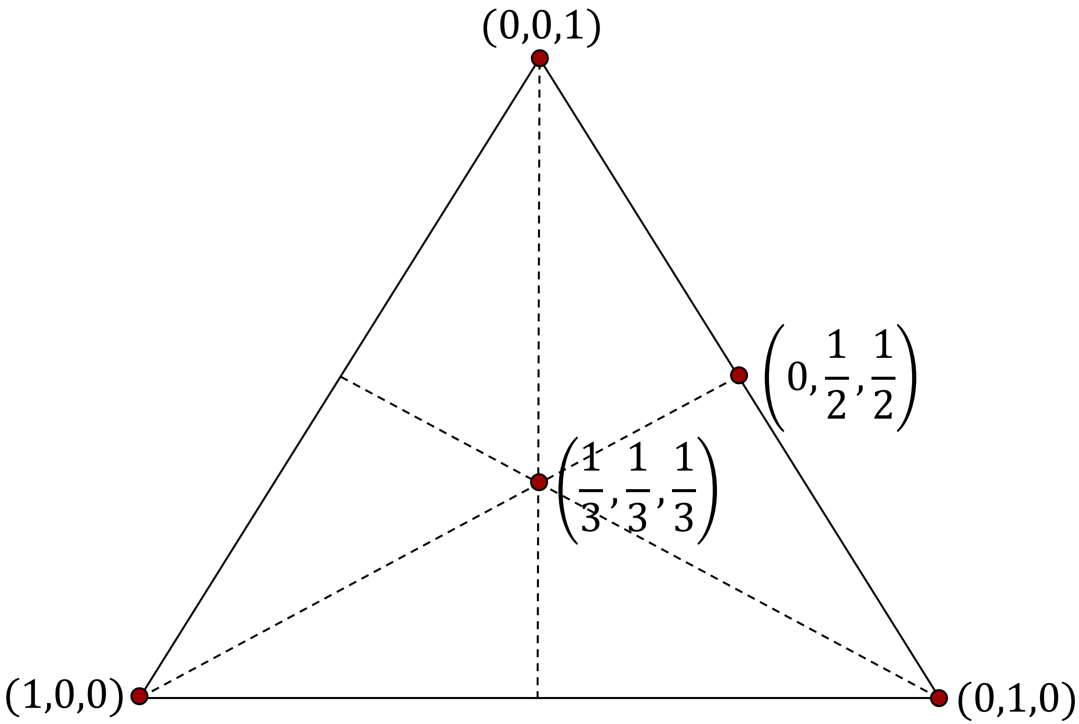
Proof of Lemma

- Start at door on boundary and walk through it
- Room is fully labeled or it has another door...
- No room visited twice
- Eventually walk into fully labeled room or back to boundary
- But #doors on boundary is odd ■



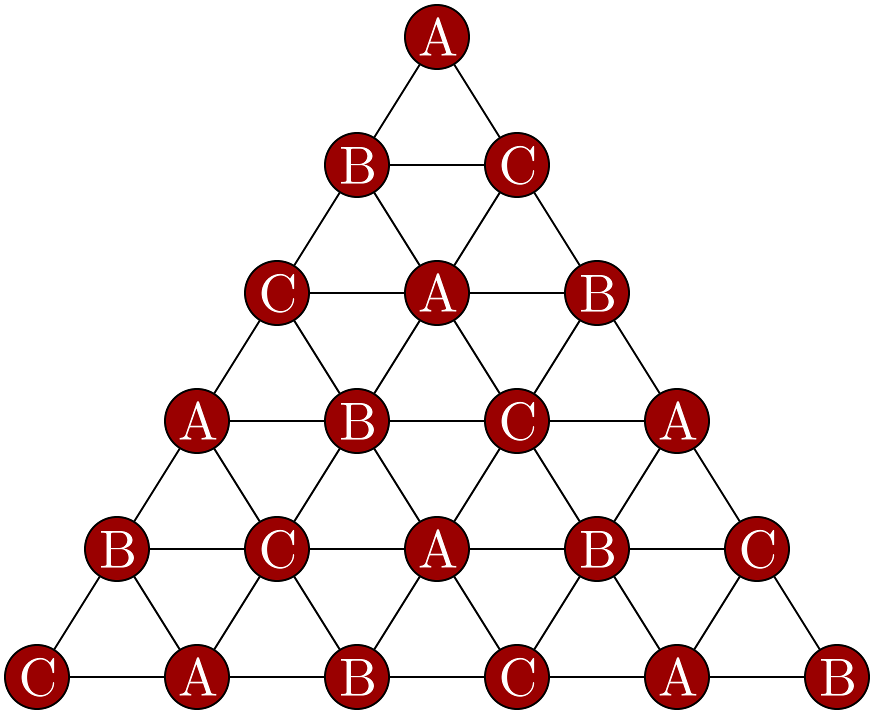
Fair rent division

- Assume there are three housemates A, B, C
- Goal is to divide rent so that each person wants different room
- Sum of prices for three rooms is 1
- Can represent possible partitions as triangle


(0,0,1)
$\left(0, \frac{1}{2}, \frac{1}{2}\right)$
$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
(1,0,0)
(0,1,0)

Fair rent division

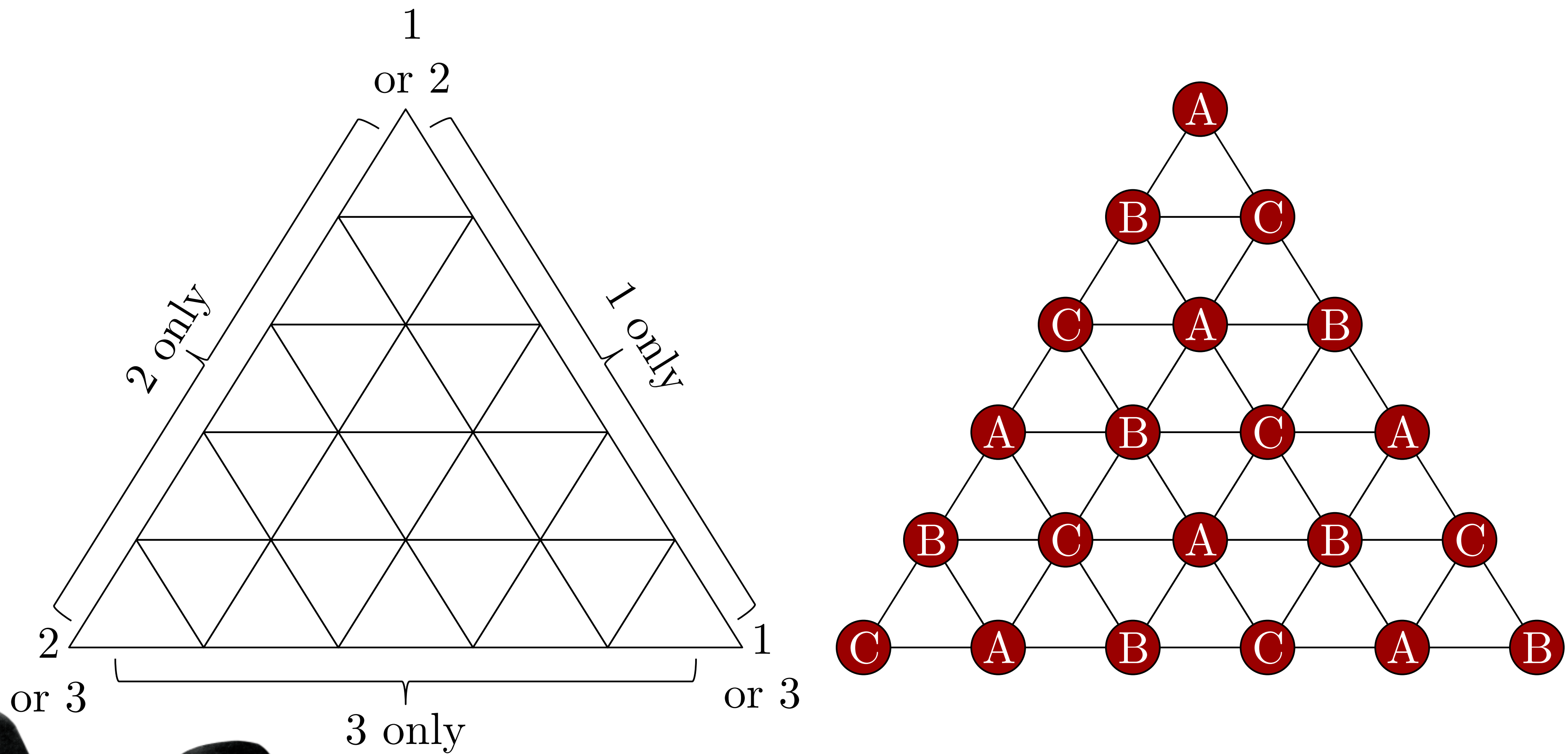
- "Triangulate" and assign "ownership" of each vertex to each of A, B, and C …
- … in a way that each elementary triangle is an ABC triangle


A
B C
C A B
A B C A
B C A B C
C A B C A B

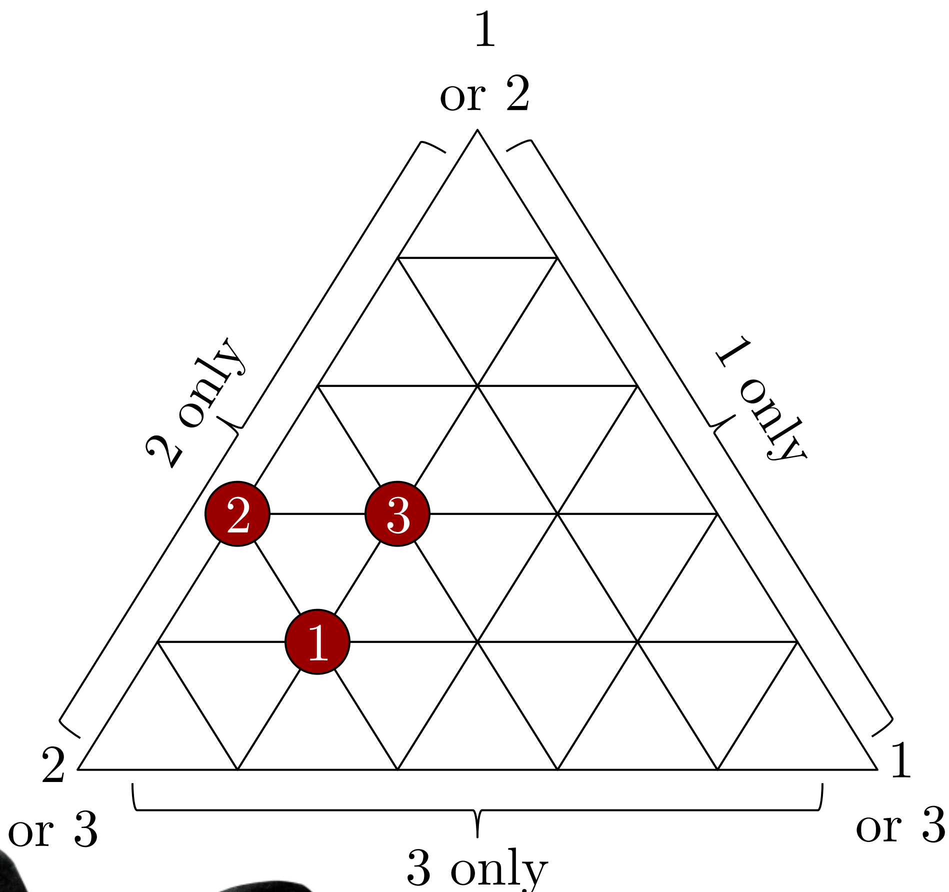
Fair rent division

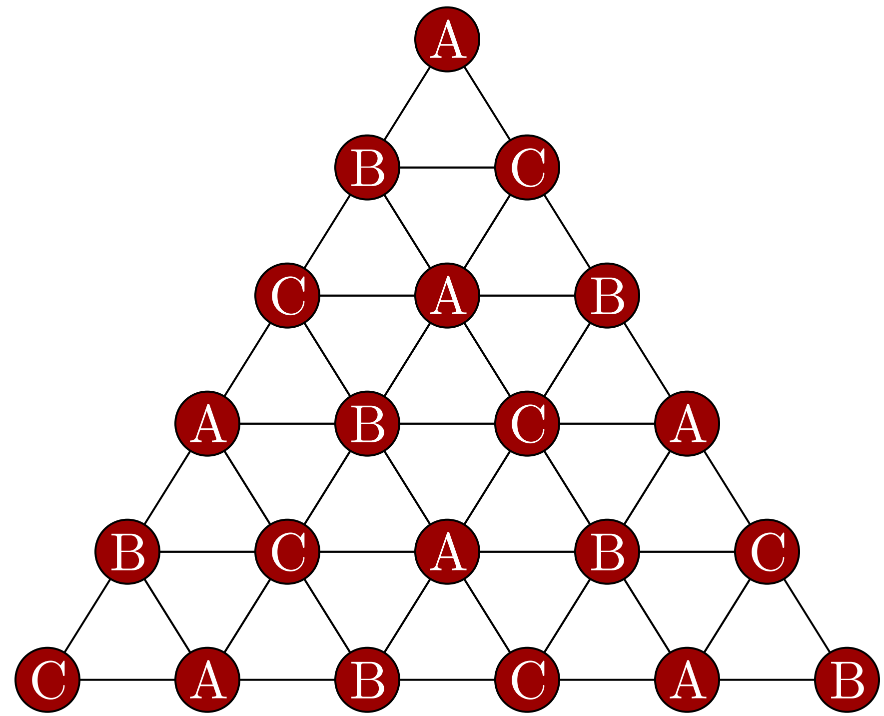
- Ask the owner of each vertex to tell us which room he prefers
- This gives a new labeling by 1, 2, 3
- Assume that a person wants a free room if one is offered to him

- Choice of rooms on edges is constrained by the free room assumption



- Sperner's lemma (variant): such a labeling must have a 123 triangle





Fair rent division

- Such a triangle is nothing but an approximately envy free allocation!
- By making the triangulation finer, we can increase accuracy
- In the limit we obtain a completely envy free allocation
- Same techniques generalize to more housemates [Su 1999]

Computational resources

- Setting: allocating multiple homogeneous resources to agents with different requirements
- Running example: shared cluster
- Assumption: agents have proportional demands for their resources (Leontief preferences)
- Example:
    - Agent has requirement (2 CPU,1 RAM) for each copy of task
    - Indifferent between allocations (4,2) and (5,2)

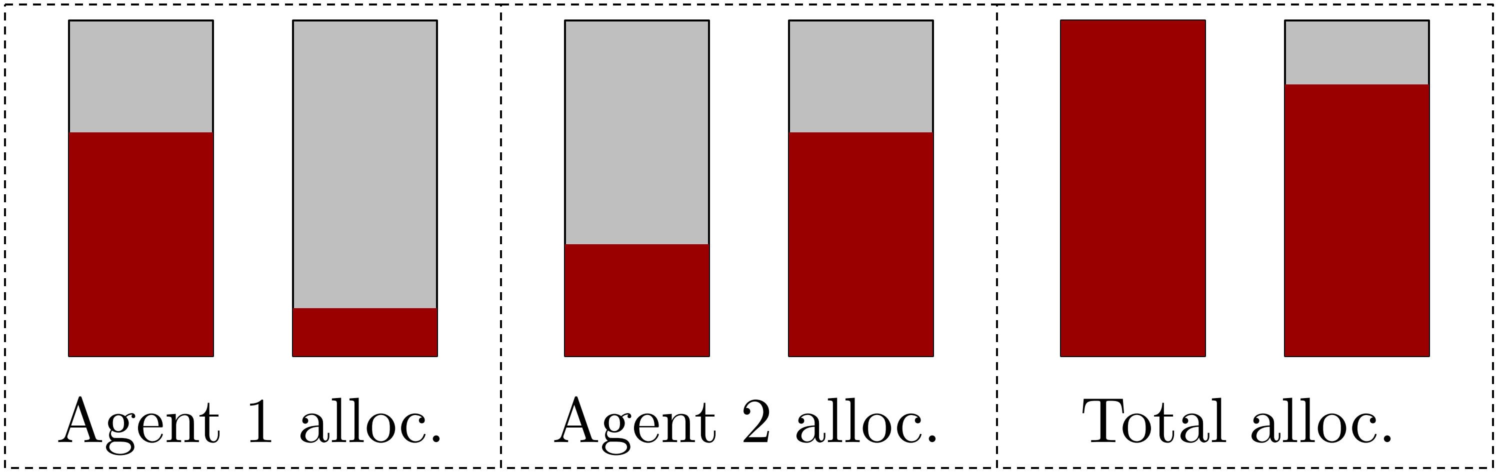
Model

- Set of players $N = \{1, \ldots, n\}$ and set of resources $R, |R| = m$
- Demand of player i is $\boldsymbol{d}_i = (d_{i1}, \ldots, d_{im})$, $0 < d_{ir} \leq 1$; $\exists r$ s.t. $d_{ir} = 1$
- Allocation $\boldsymbol{A}_i = (A_{i1}, \ldots, A_{im})$ where A_{ir} is the fraction of r allocated to i
- Preferences induced by the utility function

$$u_i(\boldsymbol{A}_i) = \min_{r \in R} A_{ir}/d_{ir}$$

Dominant resource fairness

- Dominant resource of $i = r$ s.t. $d_{ir} = 1$
- Dominant share of $i = A_{ir}$ for dominant r
- Mechanism: allocate proportionally to demands and equalize dominant shares



Formally…

- DRF finds x and allocates to i an xd_{ir} fraction of resource r:

$$\max x \text{ s.t. } \forall r \in R, \sum_{i \in N} x \cdot d_{ir} \leq 1$$

- Equivalently, $x = \dfrac{1}{\max_{r \in R} \sum_{i \in N} d_{ir}}$
- Example: $d_{11} = \frac{1}{2}; d_{12} = 1; d_{21} = 1; d_{22} = \frac{1}{6}$ then $x = \dfrac{1}{\frac{1}{2}+1} = \dfrac{2}{3}$

Axiomatic properties

- Pareto optimality (PO)
- Envy-freeness (EF)
- Proportionality (a.k.a. sharing incentives, individual rationality):

$$\forall i \in N, u_i(\boldsymbol{A}_i) \geq u_i\left(\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)\right)$$

- Strategyproofness (SP)

Properties of DRF

- An allocation $\boldsymbol{A}_i$ is non-wasteful if $\exists x$ s.t. $A_{ir} = x d_{ir}$ for all r
- If $\boldsymbol{A}_i$ is non-wasteful and $u_i(\boldsymbol{A}_i) < u_i(\boldsymbol{A}_i')$ then $A_{ir} < A_{ir}'$ for all r
- Theorem [Ghodsi et al. 2011]: DRF is PO, EF, proportional, and SP

Proof of theorem

- PO: obvious
- EF:
  - Let r be the dominant resource of i
  - $A_{ir} = x \cdot d_{ir} = x \geq x \cdot d_{jr} = A_{jr}$
- Proportionality:
  - For every r, $\sum_{i \in N} d_{ir} \leq n$
  - Therefore, $x = \frac{1}{\max_r \sum_{i \in N} d_{ir}} \geq \frac{1}{n}$

Proof of theorem

- Strategyproofness:
  - d'_{jr} are the manipulated demands; $d'_{jr} = d_{jr}$ for all $j \neq i$
  - Allocation is $A'_{jr} = x' d'_{jr}$
  - If $x' \leq x$, r is the dominant resource of i, then $A'_{ir} = x' d'_{ir} \leq x d'_{ir} \leq x d_{ir} = A_{ir}$
  - If $x' > x$, let r be the resource saturated by $\boldsymbol{A}$ ($\sum_{j \in N} x d_{jr} = 1$), then

$$A_{ir} = 1 - \sum_{j \neq i} A_{jr} = 1 - \sum_{j \neq i} x d_{jr} > 1 - \sum_{j \neq i} x' d_{jr} = 1 - \sum_{j \neq i} A'_{jr} \geq A'_{ir}$$