TRUTH JUSTICE ALGOS

Game Theory IV:
Complexity of Finding a Nash Equilibrium

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COMPUTING A NASH EQUILIBRIUM

Who cares??

If centralized, specially designed algorithms cannot find Nash equilibria, why should we expect distributed, selfish agents to naturally converge to one?

THE PROBLEM

- NASH
  - Input:
    - Number of player n.
    - An enumeration of the strategy set S_p for every player p.
    - The utility function u_p for every player.
    - An approximation requirement ϵ.
  - Output: Compute an ϵ Nash equilibrium
    - Every action that is played with positive probability is an ϵ maximizer (given the other players' strategies)
- Approximation is necessary!
  - There are games with unique irrational equilibria

HOW HARD IS IT TO COMPUTE AN EQUILIBRIUM

- NP-hard perhaps?
- What would a reduction look like?
- Typical reduction: 3SAT to Hamilton cycle
  - Take an instance I of 3SAT
  - Create an instance I' of HC
  - If I' has a Hamiltonian cycle, find a satisfying assignment for I
  - If I' doesn't have Hamiltonian cycle, conclude that there is no satisfying assignment for I

HOW HARD IS IT TO COMPUTE AN EQUILIBRIUM

- 3SAT to NASH?
  - Take an instance I of 3SAT
  - Create an instance I' of NASH
  - If I' has a MNE, find a satisfying assignment for I
  - If I' doesn't have a MNE, conclude that there is no satisfying assignment for I
- All games have a Mixed Nash Equilibrium!

HOW HARD IS IT TO COMPUTE AN EQUILIBRIUM

- What about Pure Nash?
  - Those don't always exist!
  - NP-hard! [Conitzer, Sandholm 2002]
- What about MNE with "social welfare at least x"?
  - NP-hard! [Conitzer, Sandholm 2002]
- What about just MNE?
  - Can't be NP-hard…
  - Doesn't seem to be in P either…
  - Where is it??

WHICH COMPLEXITY CLASS

NP

$\uparrow$

P

WHICH COMPLEXITY CLASS

FNP

If it's a "yes" instance, also give me the solution

FP

WHICH COMPLEXITY CLASS

FNP

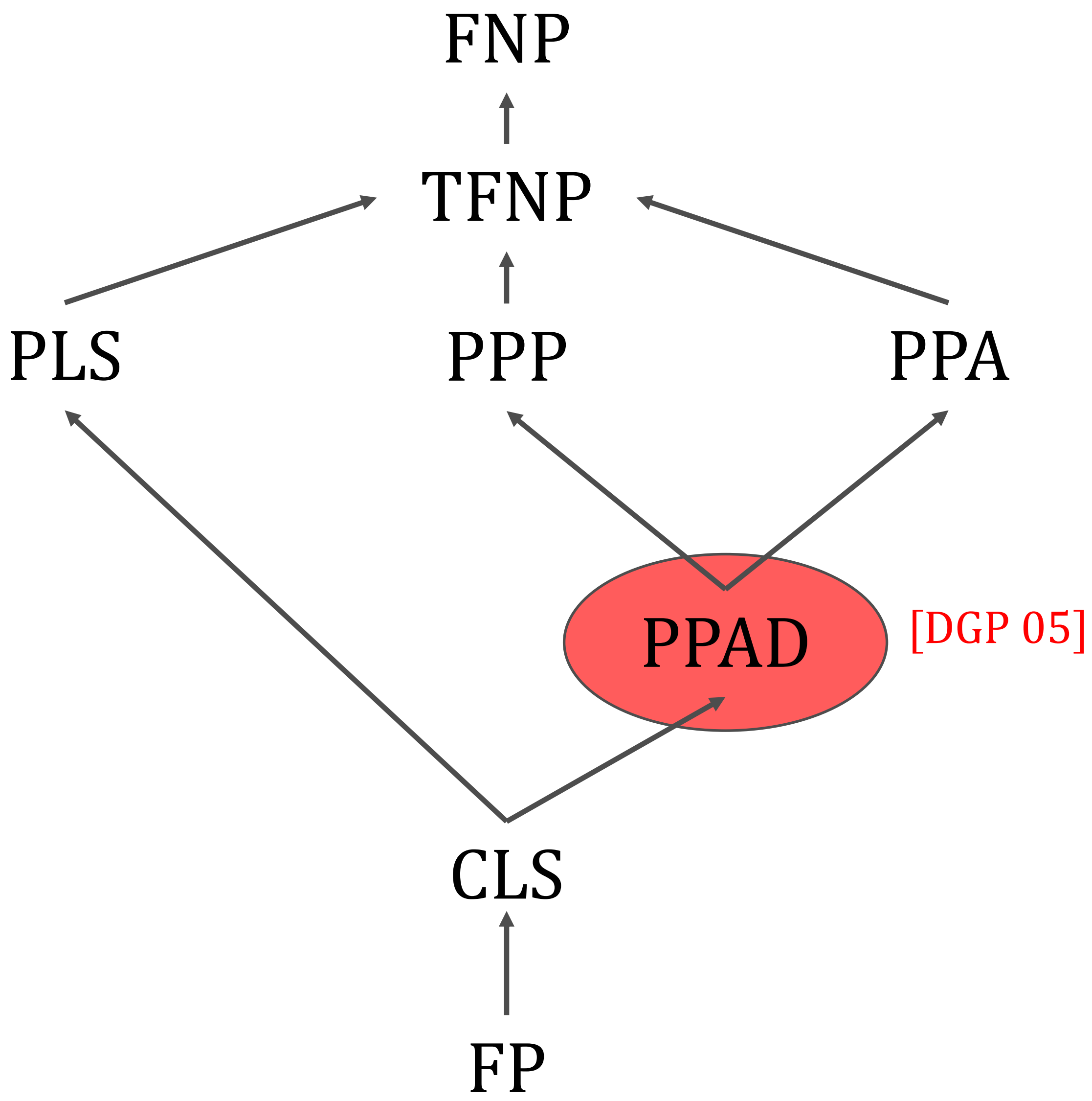
If it's a "yes" instance, also give me the solution

A "yes" instance always exists

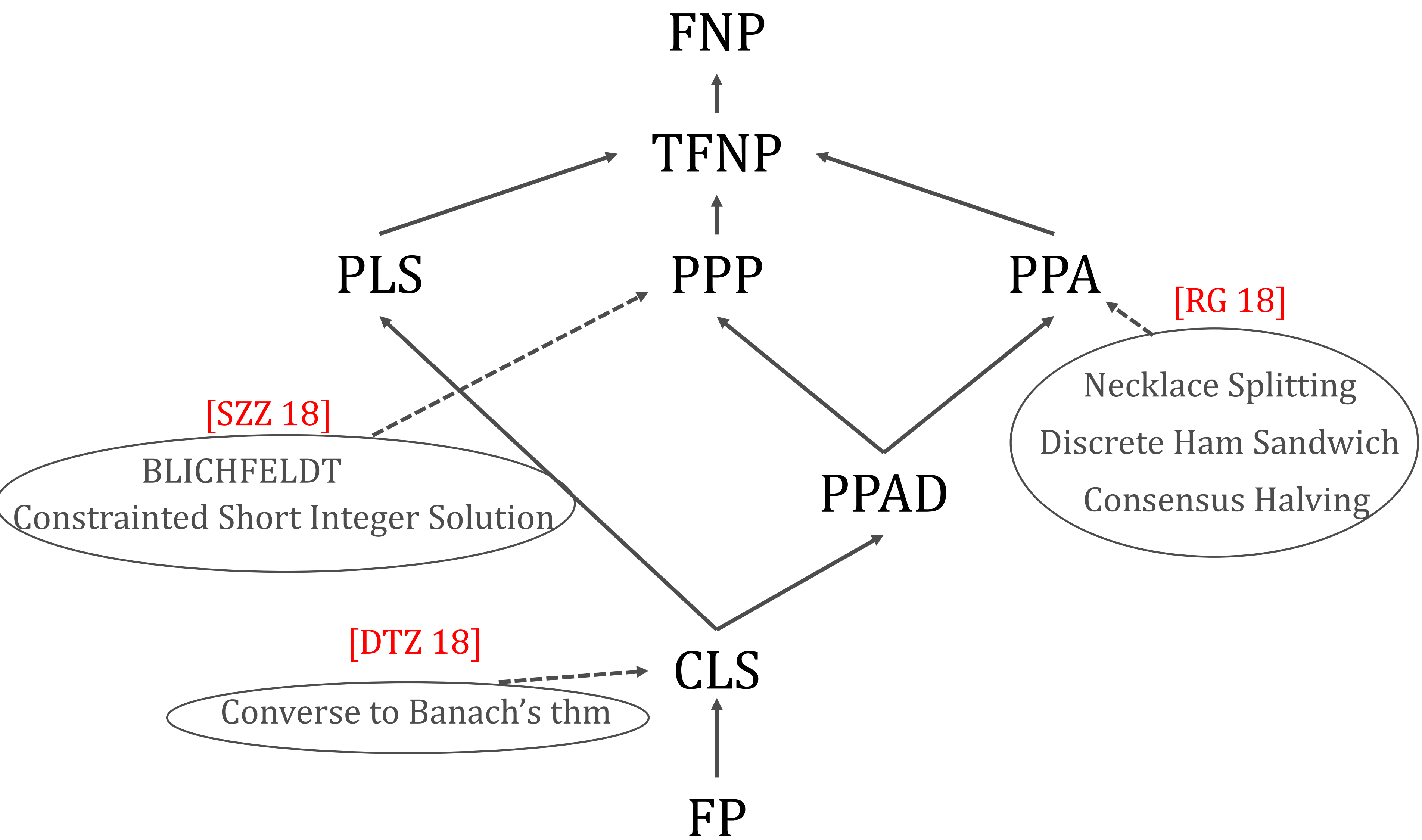
TFNP

FP

WHICH COMPLEXITY CLASS

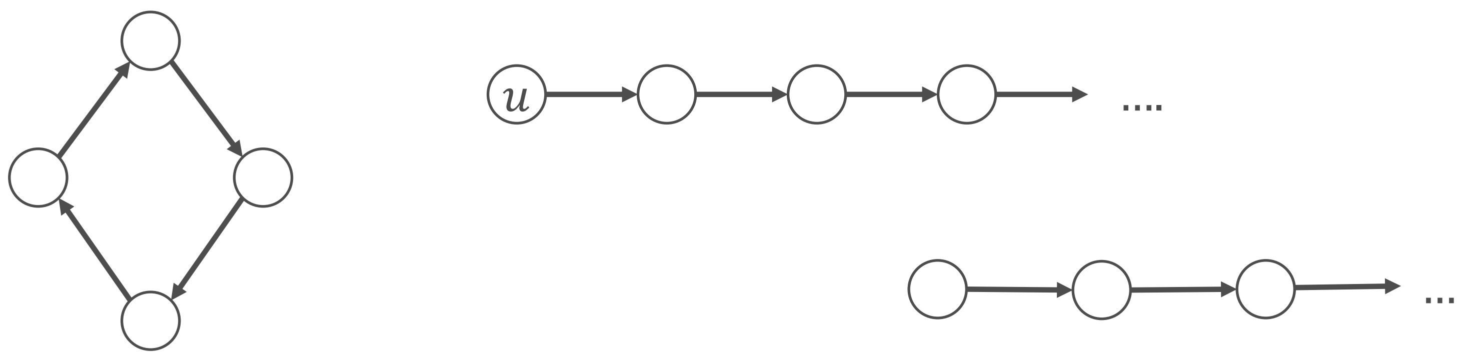


INCIDENTALLY



PPAD

- PPAD: Polynomial Parity Arguments on Directed graphs [Papadimitriou 1994]
- Input: A graph where each vertex has at most in- and out- degree at most 1. A source u.
- Goal: Find a sink or a different source!



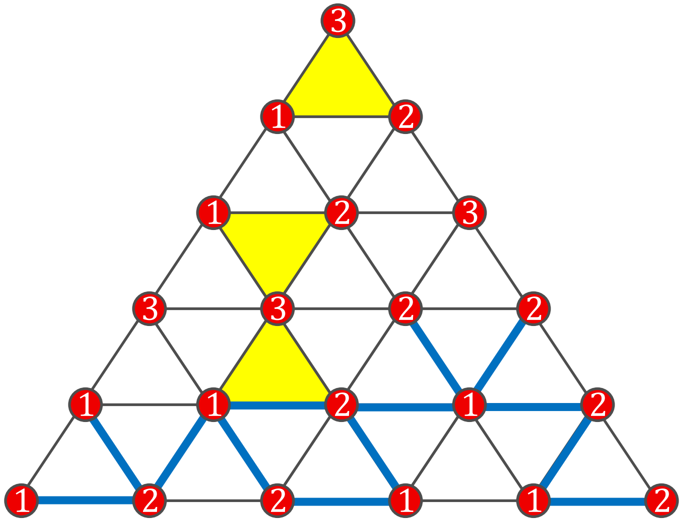
PPAD

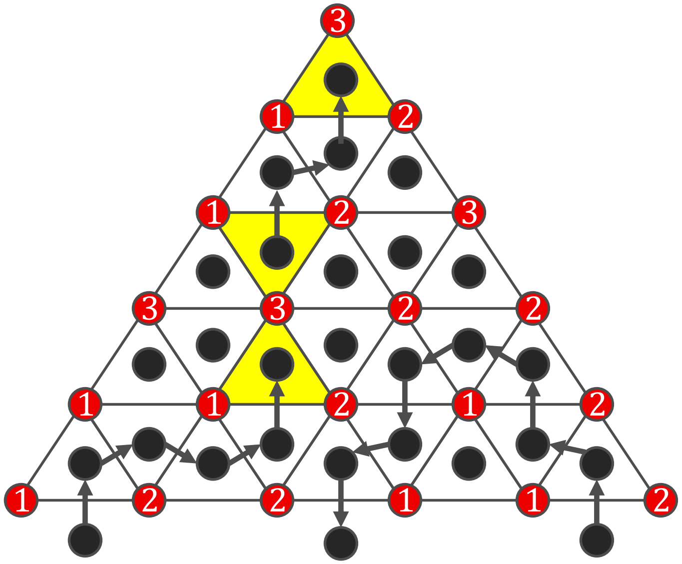
- Why not search the whole graph?
- Graph size is exponential!
- **EndOfALine**: Given two circuits S and P, with m input bits and m output bits each, such that $P(0^m) = 0^m \neq S(0^m)$, find an input $x \in \{0,1\}^m$ such that $P\big(S(x)\big) \neq \mathrm{x}$ or $S\big(P(x)\big) \neq x \neq 0^m$.
- PPAD the set of problems reducible to EndOfALine.

WHAT DOES MNE HAVE TO DO WITH ALL THIS?

- Nash's proof that every finite game has a MNE uses a fixed point theorem argument, **Brouwer's** fixed point theorem.
- The proof of Brouwer's fixed point theorem uses **Sperner's** Lemma.
- The proof of Sperner's Lemma is at its heart an exponential time **path-following** algorithm!

SPERNER'S LEMMA





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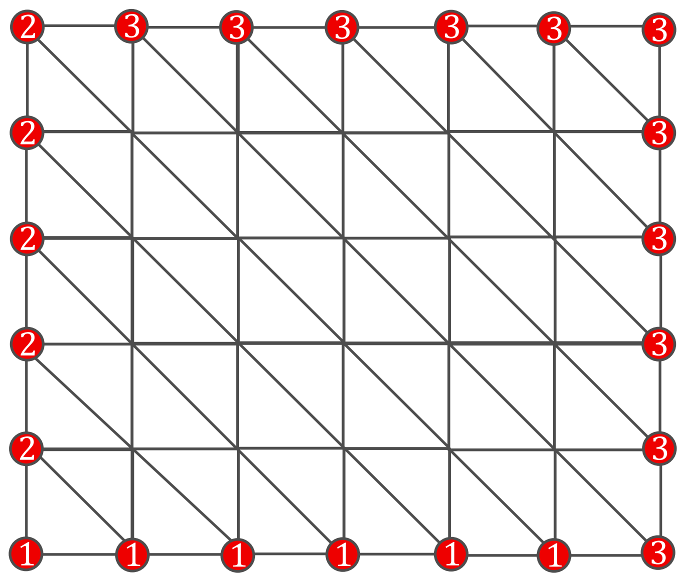
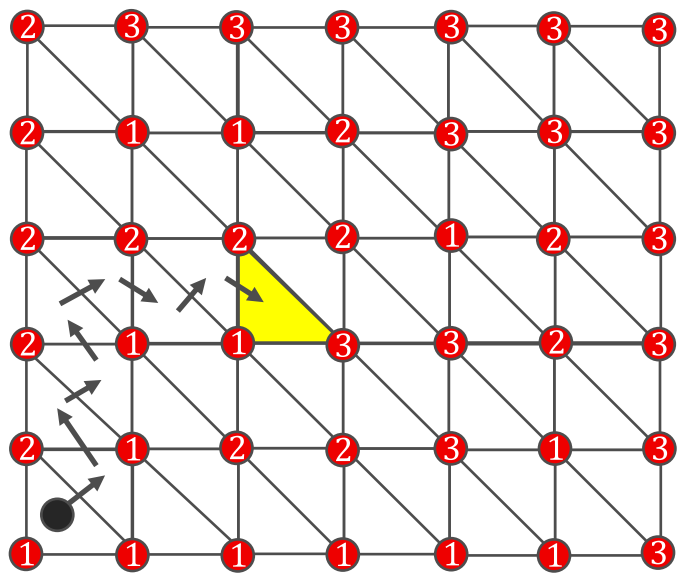


- 2D Sperner:
  - Input: The description of a poly-time Turing machine f that gives a valid coloring. $f(p) \in \{\ 0, 1, 2\ \}$, where p is a node.
  - Output: A trichromatic triangle
- 2D-Sperner ∈ PPAD
  - Obvious reduction.
- 2D-Sperner is PPAD-complete [CD 2006]

SPERNER'S LEMMA



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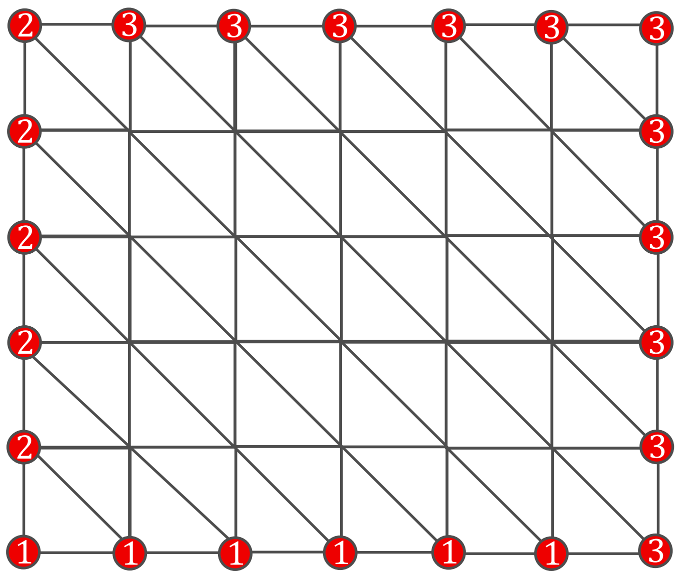
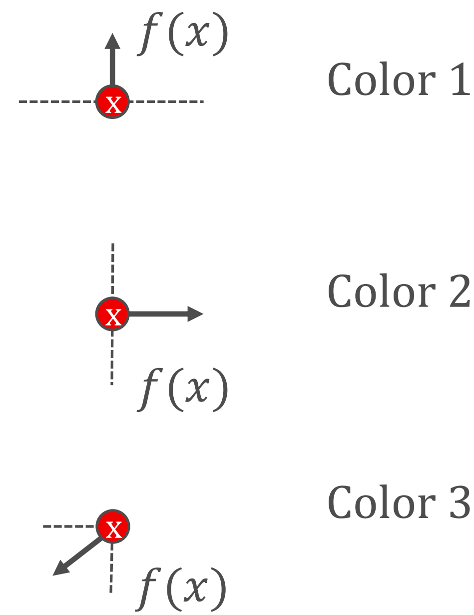
BROUWER'S FIXED POINT THEOREM

- Thm: Every continuous function f from a closed, convex and compact set C to itself has a fixed point, i.e. a point x_0 such that $f(x_0) = x_0$
- Proof (for $C =[0,1]^2$)
  - Subdivide C into tiny triangles
  - Color the edges like before.
  - For the internal nodes $x = (x_1, x_2)$:
    - If $f_2(x) \geq x_2$, color x with color 1
    - If $f_1(x) \geq x_1$, color x with color 2
    - If $f_1(x) \leq x_1$ and $f_2(x) \leq x_2$, color x with color 3
    - If more than 1 condition is met, pick an arbitrary color

BROUWER'S FIXED POINT THEOREM



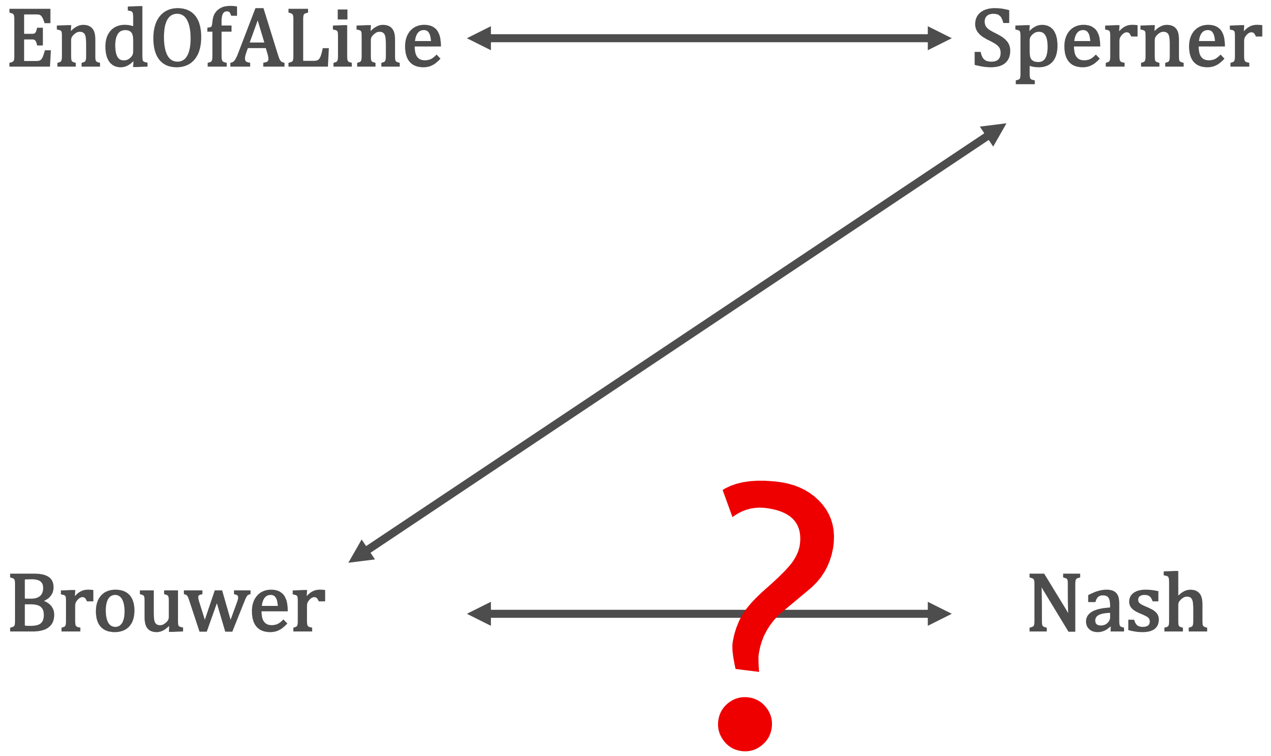


- Color 1 $= f(x)$ farther from bottom than x
- Color 2 $= f(x)$ farther from left side than x
- Color 3 $= f(x)$ farther from top and right side than x
- Trichromatic triangle (in the limit) $= f(x)$ farther from all sides than $x = x$ is a fixed point!

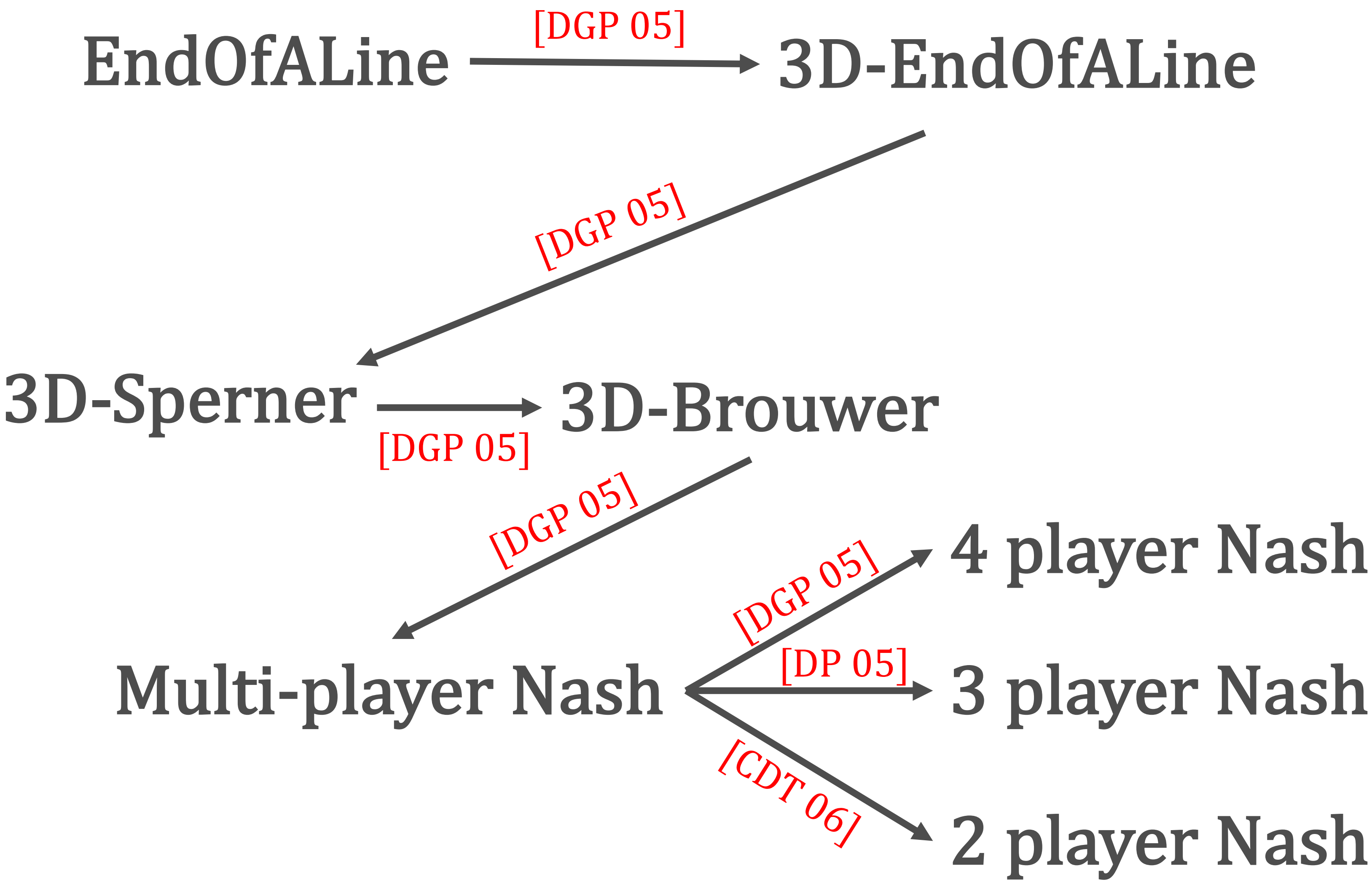
BROUWER'S FIXED POINT THEOREM

- The fixed point could be irrational!
  - We need approximation!
- Brouwer computational problem
  - Input: An algorithm that evaluates a continuous function f from $[0,1]^n$ to $[0,1]^n$. An approximation ϵ. A Lipschitz constant c that f is claimed to satisfy.
  - Output: x such that $|f(x) - x| < \epsilon$, or a violation of the assumptions
    - $A(x)$ outside $[0,1]^n$, or $|f(x) - f(y)| > c|x - y|$
- Brouwer is PPAD-complete [DGP 05]

STORY SO FAR



THE ACTUAL STORY



BROUWER →NASH?

- NASH
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BROUWER →NASH?

- Alice picks $x \in [0,1]^n$. Bob picks $y \in [0,1]^n$.
- $U_A(x, y) = -||x - y||_2^2$
- $U_B(x, y) = -||f(x) - y||_2^2$
- Claim: Equilibrium strategies must be pure.
- The only pure equilibrium is $x = y = f(x)$.
  - Why?
- Done???

POLL

Poll

What's the problem with this reduction?

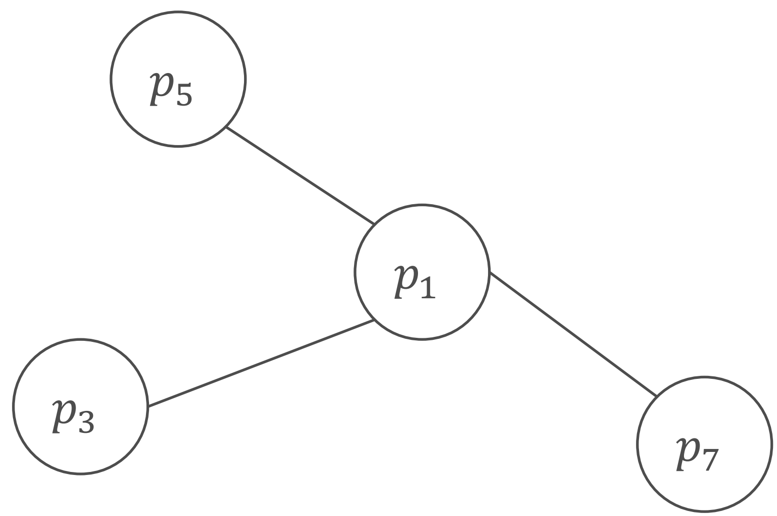
1. Too many strategies!
2. Wrong direction!
3. Those games are easy!
4. Beats me!

BROUWER →NASH?

- The computational versions of Brouwer and Sperner, as well as EndOfALine, are defined in terms of explicit circuits.
- These need to somehow be simulated in the target problem, NASH, which has no explicit circuits in its description!
- Other problems (say HC) don't have circuits either, but at least are combinatorial, which is not the case here either…

BROUWER →MULTIPLAYER NASH

- Players are nodes in a graph
- A player's payoff is only affected by her own strategy and the strategies of her neighbors



THE WHOLE STORY

- Exponential approximation is PPAD complete for 3 players [DGP 06]
- Polynomial approximation is PPAD complete for 2 player NASH [CDT 06]
- Constant approximation is PPAD complete for n players [Rubinstein 15]
- Quasi-polynomial time algorithm for ϵ approximation for 2 player [LMM 03]
- Assuming ETH for PPAD, ϵ approximation takes time $2^{\Omega(n)}$ [Rubinstein 16]

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