TRUTH JUSTICE ALGOS

Game Theory III: Positive Results

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TODAY

- Zero-sum games
- QPTAS for two players
- Exact equilibrium in exponential time for two players

ZERO SUM GAMES

- Basic definitions
  - Input: $m \times n$ matrix A
  - $A_{i,j}$ is the gain of the row player (and the loss of the column player) when the row player picks pure strategy i and the column player picks pure strategy j
  - Gain of row player for mixed strategies x, y equals $x^T A y = \sum_i \sum_j x_i A_{i,j} y_j$
  - Output: an equilibrium x, y

PURE STRATEGIES

- When the row player picks a pure strategy i, the utility she should expect is $\min_j A_{i,j}$
- So, by picking pure strategies she shouldn't be able to make more than $\max_i \min_j A_{i,j}$
- $\max_i \min_j A_{i,j} \leq \min_j \max_i A_{i,j}$
  - Why?

POLL

$$\max_i \min_j A_{i,j} \leq \min_j \max_i A_{i,j}$$

Poll X

Can this inequality be strict?

1. Yes
2. No
3. Beats me!

MIXED STRATEGIES

- When the row player picks a mixed strategy x, the utility she should expect is $\min_{y} x^T Ay$
- So, by picking mixed strategies she shouldn't be able to make more than $\max_{x} \min_{y} x^T A\, y$
- Similarly, column player shouldn't lose more than $\min_{y} \max_{x} x^T A\, y$

MINMAX THEOREM

- Theorem (Von Neumann): For every two player zero-sum game there is a value V called the value of the game such that

$$\max_x \min_y x^T A y = V = \min_y \max_x x^T A y$$

COMPUTING AN OPTIMAL STRATEGY

- Example

3	-1
-2	1

- When row picks a mixed strategy $x = (\mathrm{x}_1, \mathrm{x}_2)$ her utility is

$$\mathrm{z} = \min(3x_1 - 2x_2, -x_1 + x_2)$$

  - Should we consider mixed strategies for column?

- When column picks y loss is

$$w = \max(3y_1 - y_2, -2y_1 + y_2)$$

- Maximize z, with the constraint $x_1 + x_2 \leq 1$!
  - Issues?

TWO OPTIMAL STRATEGIES

- $\max z$
- $z \leq 3x_1 - 2x_2$
- $z \leq -x_1 + x_2$
- $x_1 + x_2 = 1$
- $x_1, x_2 \geq 0$

- $\min w$
- $w \geq 3y_1 - y_2$
- $w \geq -2y_1 + y_2$
- $y_1 + y_2 = 1$
- $y_1, y_2 \geq 0$

DUALS!!!!!!

GENERAL PROOF

- The optimization problem for the row player is
  - $\max z$,
  - $z \leq \sum_i x_i A_{i,j}$, $\forall j$
  - $\sum_i x_i = 1$
- The optimization problem for the column player is its dual!
- By strong duality we get than $z^* = w^*$, or
  - $\max_x \min_j x^T A_{(.,j)} = \min_y \max_i A_{(i,.)} y$
  - $\min_j x^T A_{(.,j)} = \min_y x^T A\, y$
  - $\max_i A_{(i,.)} y = \max_x x\, A\, y$

ϵ APPROXIMATION: NON-ZERO SUM TWO PLAYER GAMES

- Input: Two n by n matrices, A and B, for the payoffs of player 1 and player 2.
  - Assume that payoffs are in [0,1]
- Dfn: An $\epsilon - NE$ (x, y) satisfies $x^T Ay \geq \max_i A_i\, y - \epsilon$, and $y^T Bx \geq \max_i B_i x - \epsilon$.
- Goal: Find $\epsilon - NE$ in time $O(n^{\frac{\log n}{\epsilon^2}})$
- Lemma [LMM 03]:
  - There exists an $\epsilon - NE$ where each player uses a strategy that uniformly samples from a multiset (of pure strategies) of size $O(logn/\epsilon^2)$.

ϵ APPROXIMATION: NON-ZERO SUM TWO PLAYER GAMES

Proof:

- A NE (x, y) does exist.
- Thought experiment: Sample k times from x and y (with repetition)
  - We view x and y as distributions
- $X =$ multiset of pure strategies after sampling from x
- Y similarly
- $x_i^* =$ (#times pure strategy i appears in $X)/k$
- $y_i^* =$ (#times pure strategy i appears in $Y)/k$

ϵ APPROXIMATION: NON-ZERO SUM TWO PLAYER GAMES

- We want to pick k large enough so that $x^*Ay^* \geq \max_i A_i y^* - \epsilon$, and $(y^*)^T Bx^* \geq \max_i B_i x^* - \epsilon$
- It suffices if all of the following to hold:
  1. $|A_i y - A_i y^*| \leq \epsilon/3, for\ all\ i$
  2. $|B_j x - B_j x^*| \leq \epsilon/3, for\ all\ j$
  3. $|xAy - x^*Ay| \leq \epsilon/3$
  4. $|y^T Bx - (y^*)^T Bx| \leq \epsilon/3$
- $x^*Ay^* \geq x^*Ay - \epsilon/3 \geq xAy - 2\epsilon/3 \geq \max_i A_i y - 2\epsilon/3 \geq \max_i A_i y^* - \epsilon$
- Similarly, $(y^*)^T Bx^* \geq \max_i B_i x^* - \epsilon$

ϵ APPROXIMATION: NON-ZERO SUM TWO PLAYER GAMES

- How big should k be?
- Let's first try to bound $\Pr[|A_i y^* - A_i y| \geq \epsilon]$
- $y_j^* = \frac{1}{k} \sum_{\ell=1}^{k} \mathbb{I}\{\, \ell - th\ sample = j \,\}$
- $\mathbb{E}\left[y_j^*\right] = y_j$
- $\Pr\left[|y_j^* - y_j| \geq \epsilon\right] \leq e^{-2k\epsilon^2}$
- Pick $k = \frac{\log n}{\epsilon^2}$: $\Pr\left[|y_j^* - y_j| \geq \epsilon\right] \leq \frac{1}{n^2}$
- So, $\Pr\left[|A_{ij} y_j^* - A_{ij} y_j| \geq \epsilon\right] \leq \frac{1}{n^2}$
- By union bound $\Pr[|A_i y^* - A_i y| \geq \epsilon] \leq \frac{1}{n}$
- Similarly for everything else (plus union bound)

ϵ APPROXIMATION: NON-ZERO SUM TWO PLAYER GAMES

- What have we done so far…
- Sampling $k = O(\frac{\log n}{\epsilon^2})$ times from a NE gives an $\epsilon - NE\ (x^*, y^*)$
- So what?? We don't know the distribution we're supposed to sample from
- Every probability in x^* and y^* is a multiple of $\frac{1}{k}$
- How many such x^* exist?
  - $\binom{n+k-1}{n-1} \approx n^k = n^{\frac{logn}{\epsilon^2}}$
- Try all of them!

EXACT EQUILIBRIA FOR 2 PLAYERS: LEMKE-HOWSON

- Focus on symmetric n by n games $B = A$
- Those have symmetric equilibria
- Consider the following polytope

$$\sum_{j=1}^{n} A_{i,j} x_j \leq 1\,, for\ all\ i$$

$$x_j \geq 0, for\ all\ j$$

- $2n$ inequalities, n dimensions
- A strategy i is ``represented'' at a corner x if at least one of the following holds: (1) $x_i = 0$, (2) $A_i x = 1$

LEMKE-HOWSON

- Lemma: If every strategy is represented at some corner x (expect the origin), then $(\frac{x}{|x|_1}, \frac{x}{|x|_1})$ a NE.

Proof:

1. $x_i = 0 \rightarrow i$ is not played

2. $x_i > 0 \rightarrow A_i x = 1 \rightarrow A_i x \geq A_j x$ for all j (recall the constraints of the polytope)

$$\rightarrow e_i^T A \frac{x}{|x|_1} \geq e_j^T A \frac{x}{|x|_1} \text{ for all } j$$

i is either not played or is a best response to $\frac{x}{|x|_1}$.

LEMKE-HOWSON

- Start at the origin
- Assuming non-degeneracy there are n adjacent corners.
- Pick a constraint to untighten, say $x_1 = 0$
- Keep all other constraints tight and pivot until one of the $A_j x \leq 1$ constraint hits
- If not at NE, there must be a strategy that is ``represented'' by two constraints
- This gives two neighbors (by relaxing each of the two constraints), one of them is the previous corner
- Go to the other one
- Repeat
- At most $\binom{2n}{n}$ vertices, so this will terminate (in exponential time)
  - Just need to prove that we don't go in circles

LEMKE-HOWSON

$$R = \begin{pmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{pmatrix}$$