TRUTH JUSTICE ALGOS

Game Theory II: Price of Anarchy

Teachers: Ariel Procaccia (this time) and Alex Psomas

BACK TO PRISON

- The only Nash equilibrium in Prisoner's dilemma is bad; but how bad is it?
- **Objective function:** social cost = sum of costs
- NE is six times worse than the optimum
- We can make this arbitrarily bad

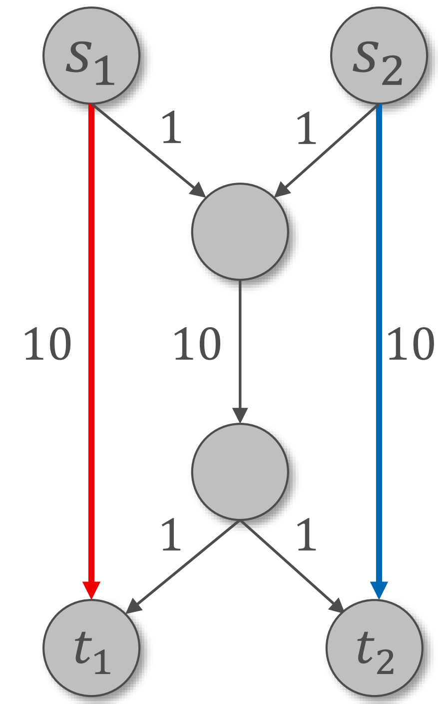
-1,-1	-9,0
0,-9	-6,-6

ANARCHY AND STABILITY

- Fix a class of games, an objective function, and an equilibrium concept
- The price of anarchy (stability) is the worst-case ratio between the worst (best) objective function value of an equilibrium of the game, and that of the optimal solution
- In this lecture:
  - Objective function = social cost
  - Equilibrium concept = Nash equilibrium

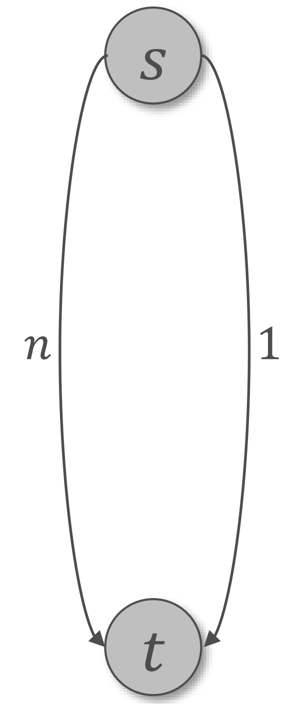
EXAMPLE: COST SHARING

- n players in weighted directed graph G
- Player i wants to get from s_i to t_i; strategy space is $s_i \rightarrow t_i$ paths
- Each edge e has cost c_e
- Cost of edge is split between all players using edge
- Cost of player is sum of costs over edges on path



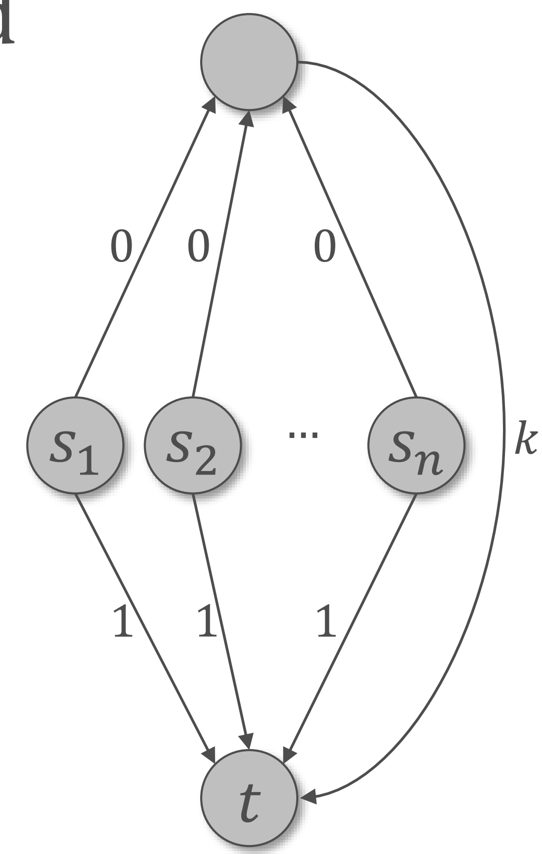
EXAMPLE: COST SHARING

- With n players, the example on the right has a NE with social cost n
- Optimal social cost is 1
- It follows that the price of anarchy of cost sharing games is at least n
- It is easy to see that the price of anarchy of cost sharing games is at most n — why?



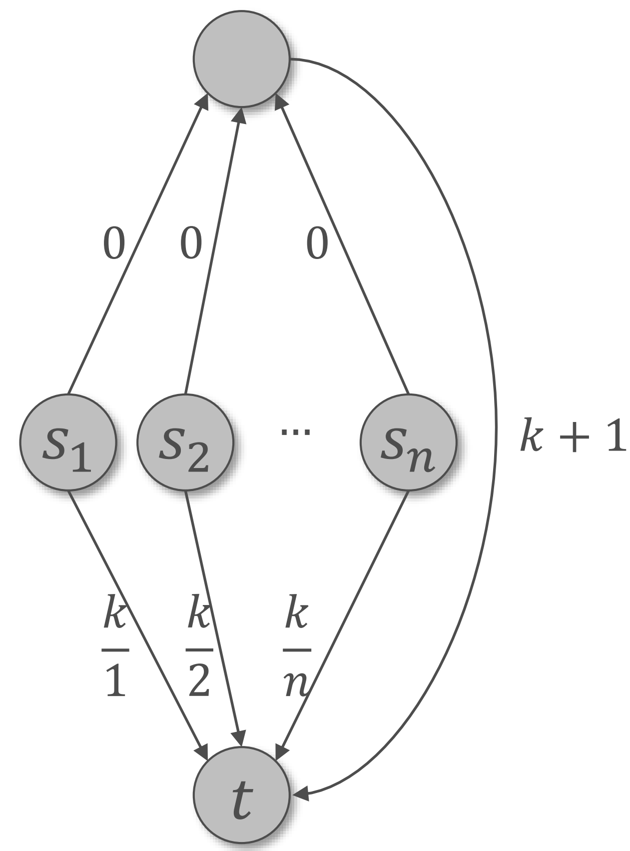
EXAMPLE: COST SHARING

- Think of the 1 edges as cars, and the k edge as mass transit
- Bad Nash equilibrium with cost n
- Good Nash equilibrium with cost k
- Now let's modify the example…



EXAMPLE: COST SHARING

- $\text{OPT} = k + 1$
- Only equilibrium has cost $k \cdot H(n)$
- Therefore, the price of stability of cost sharing games is at least $\Omega(\log n)$
- We will show that the price of stability is $\Theta(\log n)$



POTENTIAL GAMES

- A game is an exact potential game if there exists a function $\Phi: \prod_{i=1}^{n} S_i \to \mathbb{R}$ such that for all $i \in N$, for all $\boldsymbol{s} \in \prod_{i=1}^{n} S_i$, and for all $s_i' \in S_i$,
$\mathrm{cost}_i(s_i', \boldsymbol{s}_{-i}) - \mathrm{cost}_i(\boldsymbol{s}) = \Phi(s_i', \boldsymbol{s}_{-i}) - \Phi(\boldsymbol{s})$
- The existence of an exact potential function implies the existence of a pure Nash equilibrium — why?

POTENTIAL GAMES

- **Theorem:** the cost sharing game is an exact potential game
- **Proof:**
  - Let $n_e(\boldsymbol{s})$ be the number of players using e under $\boldsymbol{s}$
  - Define the potential function

$$\Phi(\boldsymbol{s}) = \sum_{e} \sum_{k=1}^{n_e(\boldsymbol{s})} \frac{c_e}{k}$$

  - If player changes paths, pays $\frac{c_e}{n_e(\boldsymbol{s})+1}$ for each new edge, gets $\frac{c_e}{n_e(\boldsymbol{s})}$ for each old edge, so $\Delta\mathrm{cost}_i = \Delta\Phi$ ■

POTENTIAL GAMES

- **Theorem:** The cost of stability of cost sharing games is $O(\log n)$
- **Proof:**
    - It holds that
    $$\text{cost}(\boldsymbol{s}) \leq \Phi(\boldsymbol{s}) \leq H(n) \cdot \text{cost}(\boldsymbol{s})$$
    - Take a strategy profile $\boldsymbol{s}$ that minimizes Φ
    - $\boldsymbol{s}$ is an NE
    - $\text{cost}(\boldsymbol{s}) \leq \Phi(\boldsymbol{s}) \leq \Phi(\text{OPT}) \leq H(n) \cdot \text{cost}(\text{OPT})$ ■

COST SHARING SUMMARY

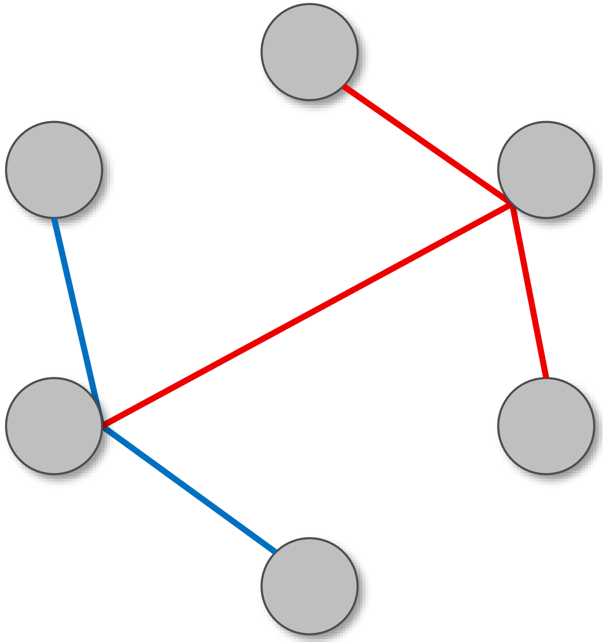
- **Upper bounds:** ∀cost sharing game,
  - PoA: ∀NE $\boldsymbol{s}$,
    $$\text{cost}(\boldsymbol{s}) \leq n \cdot \text{cost}(\text{OPT})$$
  - PoS: ∃NE $\boldsymbol{s}$ s.t.
    $$\text{cost}(\boldsymbol{s}) \leq H(n) \cdot \text{cost}(\text{OPT})$$
- **Lower bounds:** ∃cost sharing game s.t.
  - PoA: ∃NE $\boldsymbol{s}$ s.t.
    $$\text{cost}(\boldsymbol{s}) \geq n \cdot \text{cost}(\text{OPT})$$
  - PoS: ∀NE $\boldsymbol{s}$,
    $$\text{cost}(\boldsymbol{s}) \geq H(n) \cdot \text{cost}(\text{OPT})$$

NETWORK FORMATION GAMES

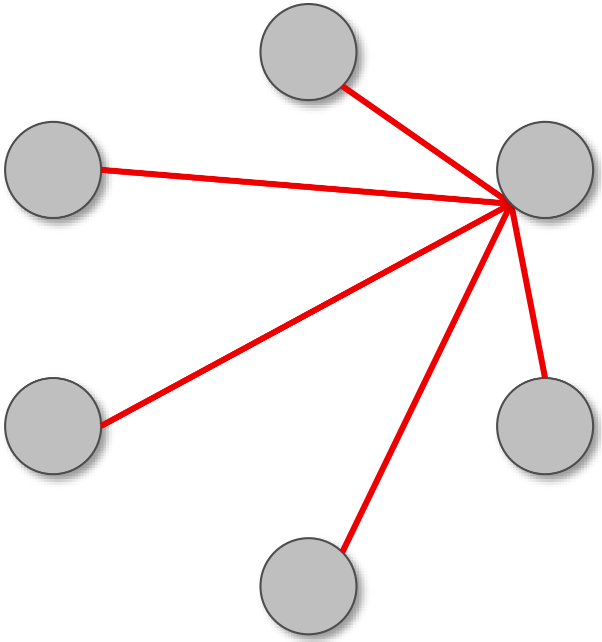
- Each player is a vertex v
- Strategy of v: set of undirected edges to build that touch v
- Strategy profile $\boldsymbol{s}$ induces undirected graph $G(\boldsymbol{s})$
- Cost of building any edge is α
- $\text{cost}_v(\boldsymbol{s}) = \alpha n_v(\boldsymbol{s}) + \sum_u d(u,v)$, where $n_v =$ #edges bought by v, d is shortest path in #edges
- $\text{cost}(\boldsymbol{s}) = \sum_{u \neq v} d(u,v) + \alpha|E|$

EXAMPLE: NETWORK FORMATION

NE with $\alpha = 3$



Suboptimal



Optimal

EXAMPLE: NETWORK FORMATION

- **Lemma:** If $\alpha \geq 2$ then any star is optimal, and if $\alpha \leq 2$ then a complete graph is optimal
- **Proof:**
  - Suppose $\alpha \leq 2$, and consider any graph that is not complete
  - Adding an edge will decrease the sum of distances by at least 2, and costs only α
  - Suppose $\alpha \geq 2$ and the graph contains a star, so the diameter is at most 2; deleting a non-star edge increases the sum of distances by at most 2, and saves α ■

EXAMPLE: NETWORK FORMATION

Poll 1

For which values of α is any star a NE, and for which is any complete graph a NE?

1. $\alpha \geq 1, \alpha \leq 1$
2. $\alpha \geq 2, \alpha \leq 1$
3. $\alpha \geq 1$, none
4. $\alpha \geq 2$, none

- **Theorem:**
  1. If $\alpha \geq 2$ or $\alpha \leq 1$, $\mathrm{PoS} = 1$
  2. For $1 < \alpha < 2$, $\mathrm{PoS} \leq 4/3$

PROOF OF THEOREM

- Part 1 is immediate from the lemma and poll
- For $1 < \alpha < 2$, the star is a NE, while OPT is a complete graph
- Worst case ratio when $\alpha \to 1$:

$$\frac{2n(n-1) - 2(n-1) + (n-1)}{n(n-1) + n(n-1)/2}$$

$$= \frac{4n^2 - 6n + 2}{3n^2 - 3n} < \frac{4}{3} \quad \blacksquare$$

EXAMPLE: NETWORK CREATION

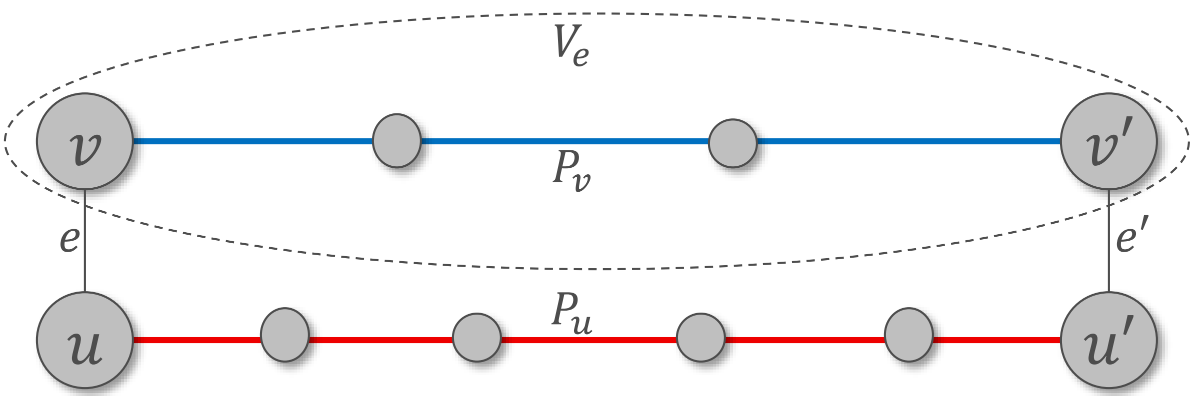
- Theorem [Fabrikant et al. 2003]: The price of anarcy of network creation games is $O(\sqrt{\alpha})$
- Lemma: If $\boldsymbol{s}$ is a Nash equilibrium that induces a graph of diameter d, then $\text{cost}(\boldsymbol{s}) \leq O(d) \cdot \text{OPT}$

PROOF OF LEMMA

- $\mathrm{OPT} = \Omega(\alpha n + n^2)$
  - Buying a connected graph costs at least $(n-1)\alpha$
  - There are $\Omega(n^2)$ distances
- Distance costs $\leq dn^2 \Rightarrow$ focus on edge costs
- There are at most $n - 1$ cut edges $\Rightarrow$ focus on noncut edges

PROOF OF LEMMA

- Claim: Let $e = (u, v)$ be a noncut edge, then the distance $d(u, v)$ with e deleted $\leq 2d$
  - V_e = set of nodes s.t. the shortest path from u uses e
  - Figure shows shortest path avoiding e, $e' = (u', v')$ is the edge on the path entering V_e
  - P_u is the shortest path from u to $u' \Rightarrow |P_u| \leq d$
  - $|P_v| \leq d - 1$ as $P_v \cup \{e\}$ is shortest path from u to v' ■



PROOF OF LEMMA

- Claim: There are $O(nd/\alpha)$ noncut edges paid for by any vertex u
  - Let $e = (u, v)$ be an edge paid for by u
  - By previous claim, deleting e increases distances from u by at most $2d|V_e|$
  - G is an equilibrium $\Rightarrow \alpha \leq 2d|V_e| \Rightarrow |V_e| \geq \alpha/2d$
  - n vertices overall $\Rightarrow$ can't be more than $2nd/\alpha$ sets V_e ■

PROOF OF LEMMA

- $O(nd/\alpha)$ noncut edges per vertex
- $O(nd)$ total payment for these per vertex
- $O(n^2d)$ overall ■

PROOF OF THEOREM

- By lemma, it is enough to show that the diameter at a NE $\leq 2\sqrt{\alpha}$
- Suppose $d(u, v) \geq 2k$ for some k
- By adding the edge (u, v), u pays α and improves distance to second half of the $u \to v$ shortest path by
$$(2k - 1) + (2k - 3) + \cdots + 1 = k^2$$
- If
$$\alpha < k^2 \leq \left(\frac{d(u, v)}{2}\right)^2 \Rightarrow d(u, v) > 2\sqrt{\alpha}$$
then it is beneficial to add edge — contradiction ■