TRUTH JUSTICE ALGOS

Fair Division IV: Rent Division

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THE WHINING PHILOSOPHERS PROBLEM

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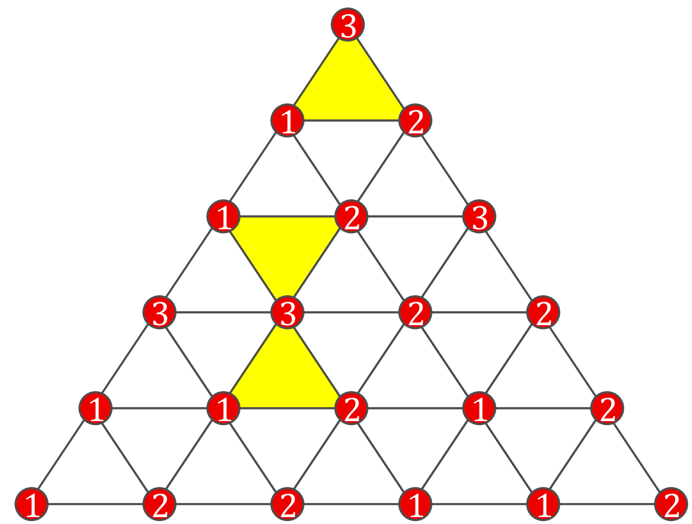
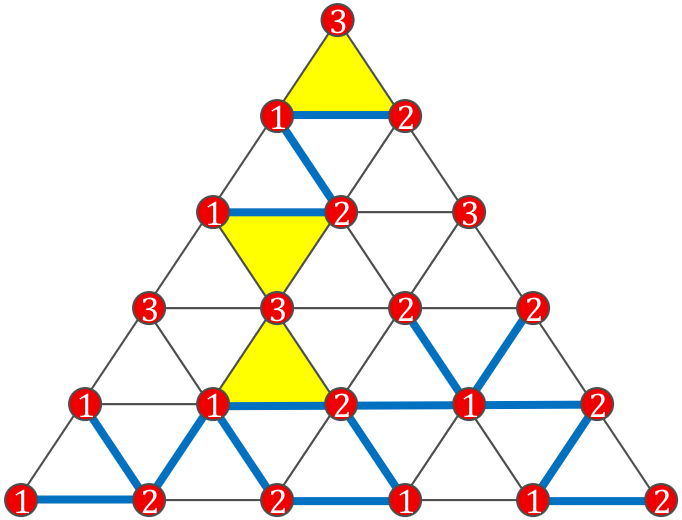
SPERNER'S LEMMA

- Triangle T partitioned into elementary triangles
- Label vertices by {1,2,3} using Sperner labeling:
  - Main vertices are different
  - Label of vertex on an edge (i, j) of T is i or j
- Lemma: Any Sperner labeling contains at least one fully labeled elementary triangle



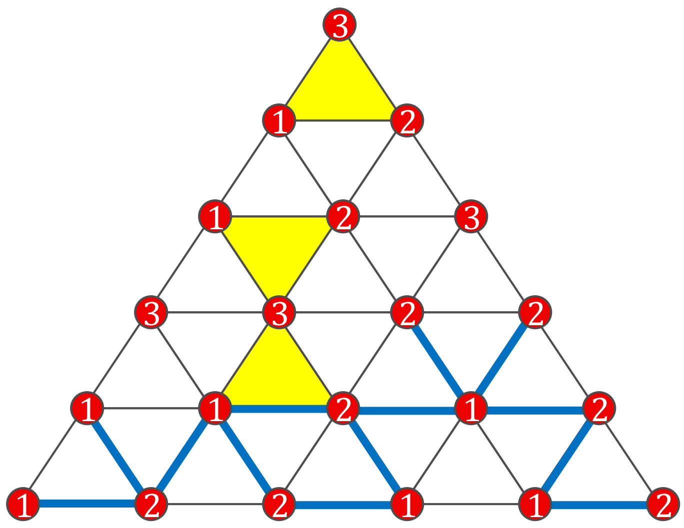
PROOF OF LEMMA

- Doors are 12 edges
- Rooms are elementary triangles
- #doors on the boundary of T is odd
- Every room has ≤ 2 doors; one door iff the room is 123



PROOF OF LEMMA

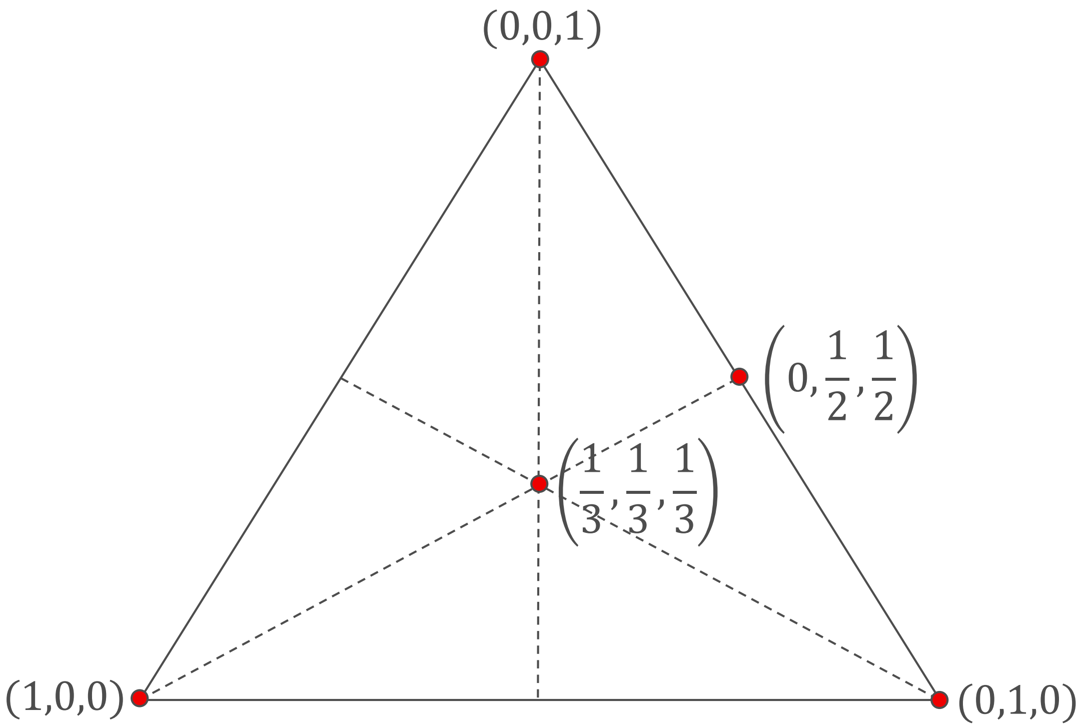
- Start at door on boundary and walk through it
- Room is fully labeled or it has another door...
- No room visited twice
- Eventually walk into fully labeled room or back to boundary
- But #doors on boundary is odd ■



THE MODEL

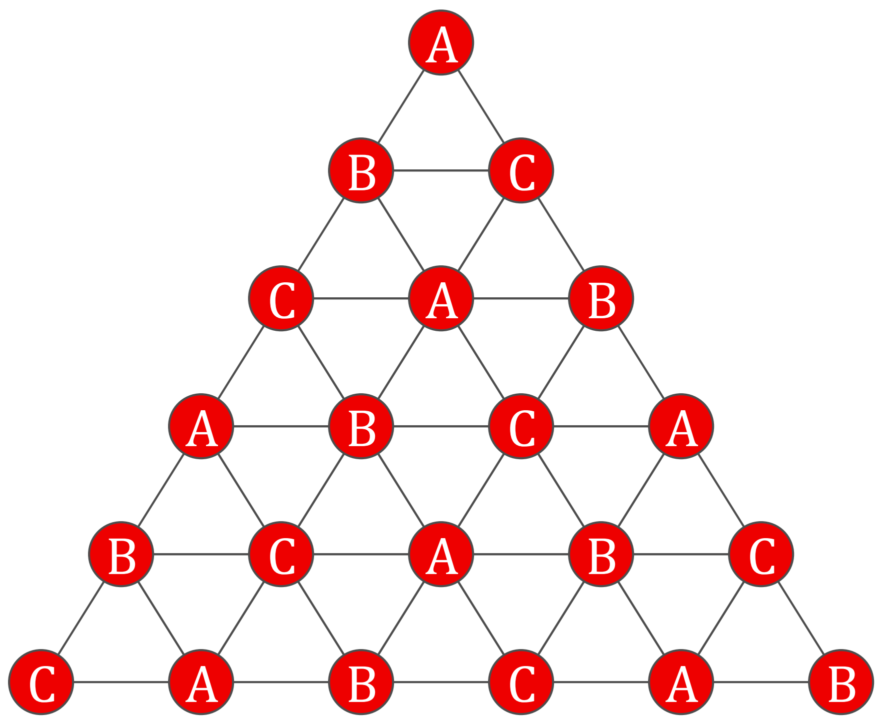
- Assume there are three players A, B, C
- Goal is to assign the rooms and divide the rent in a way that is envy free: each player wants a different room at the given prices
- Sum of prices for three rooms is 1
- Theorem [Su 99]: An envy-free solution always exists under some assumptions

PROOF OF THEOREM



PROOF OF THEOREM

- “Triangulate” and assign “ownership” of each vertex to each of A, B, and C, in a way that each elementary triangle is an ABC triangle

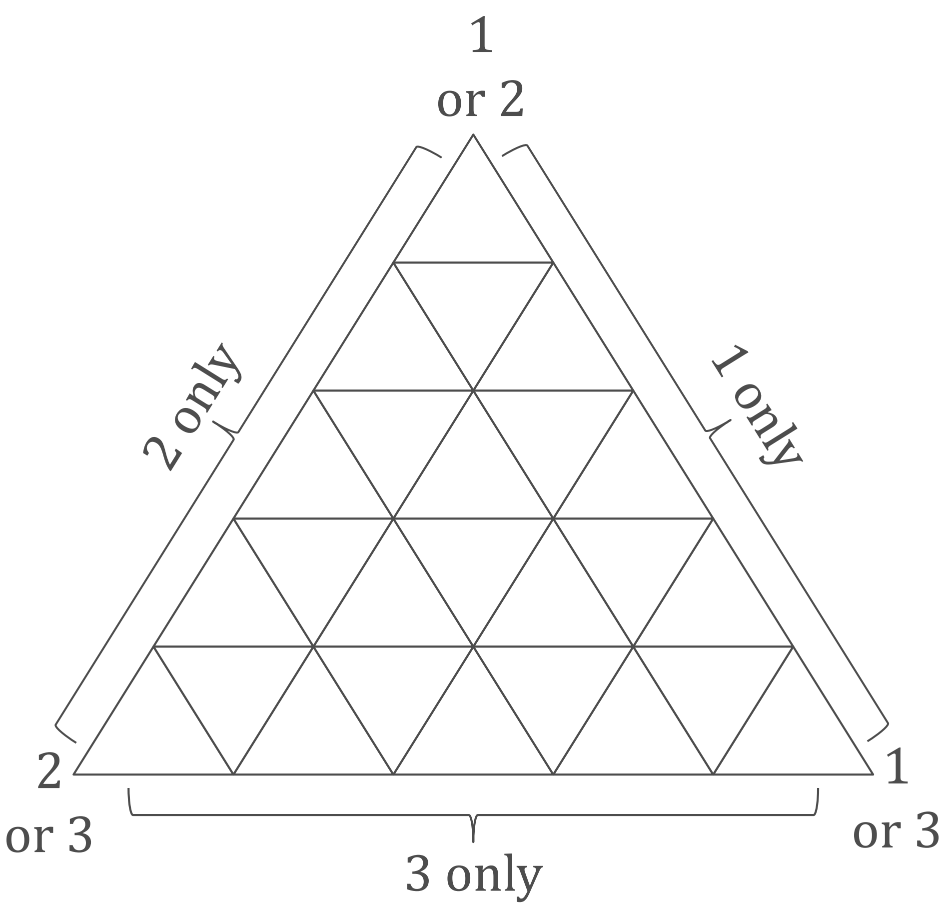


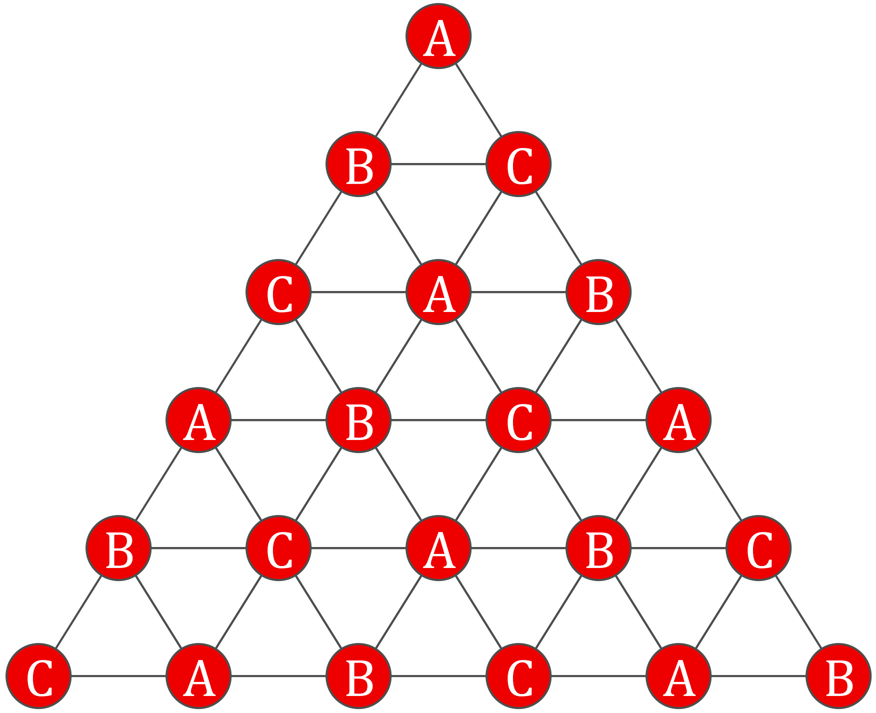
PROOF OF THEOREM

- Ask the owner of each vertex to tell us which room he prefers
- This gives a new labeling by 1, 2, 3
- Assume that a person wants a free room if one is offered to him

PROOF OF THEOREM

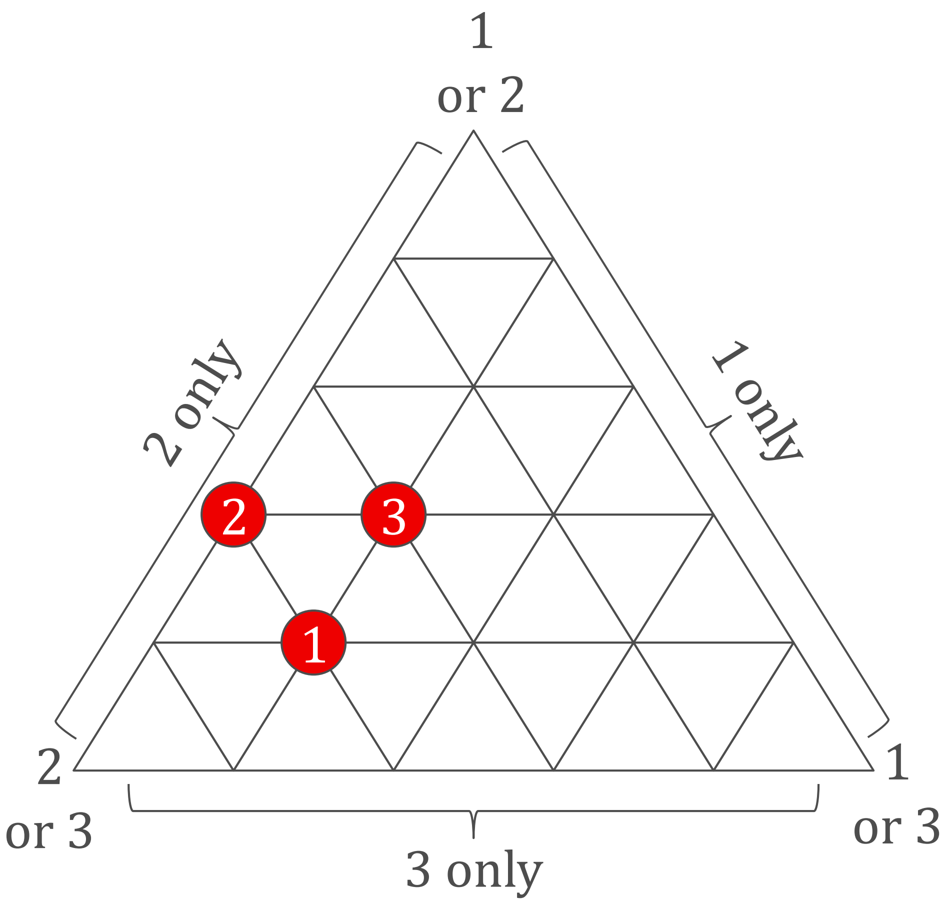
- Choice of rooms on edges is constrained by free room assumption

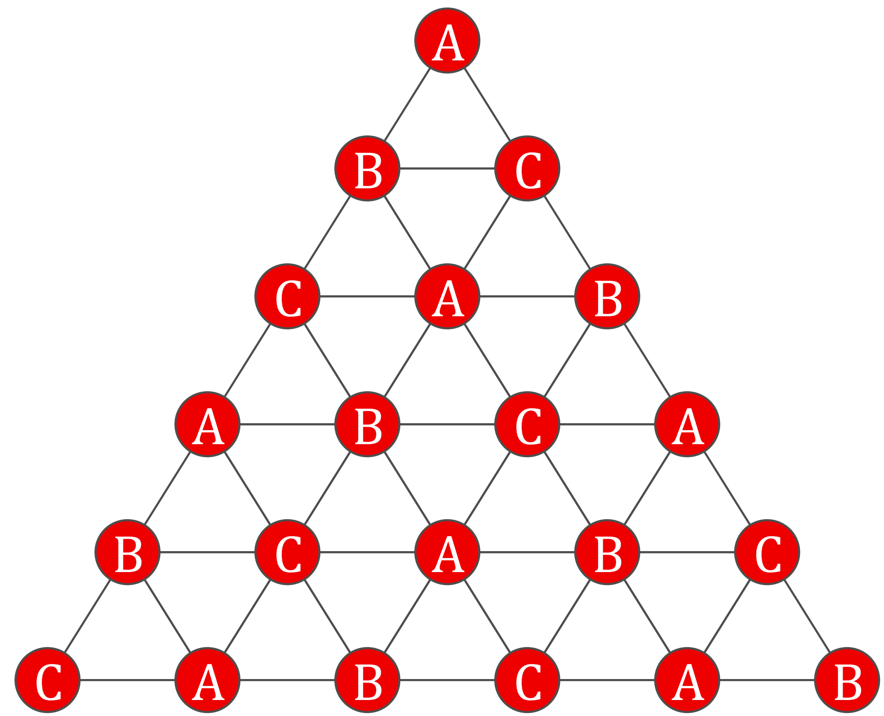




PROOF OF THEOREM

- Sperner's lemma (variant): such a labeling must have a 123 triangle





PROOF OF THEOREM

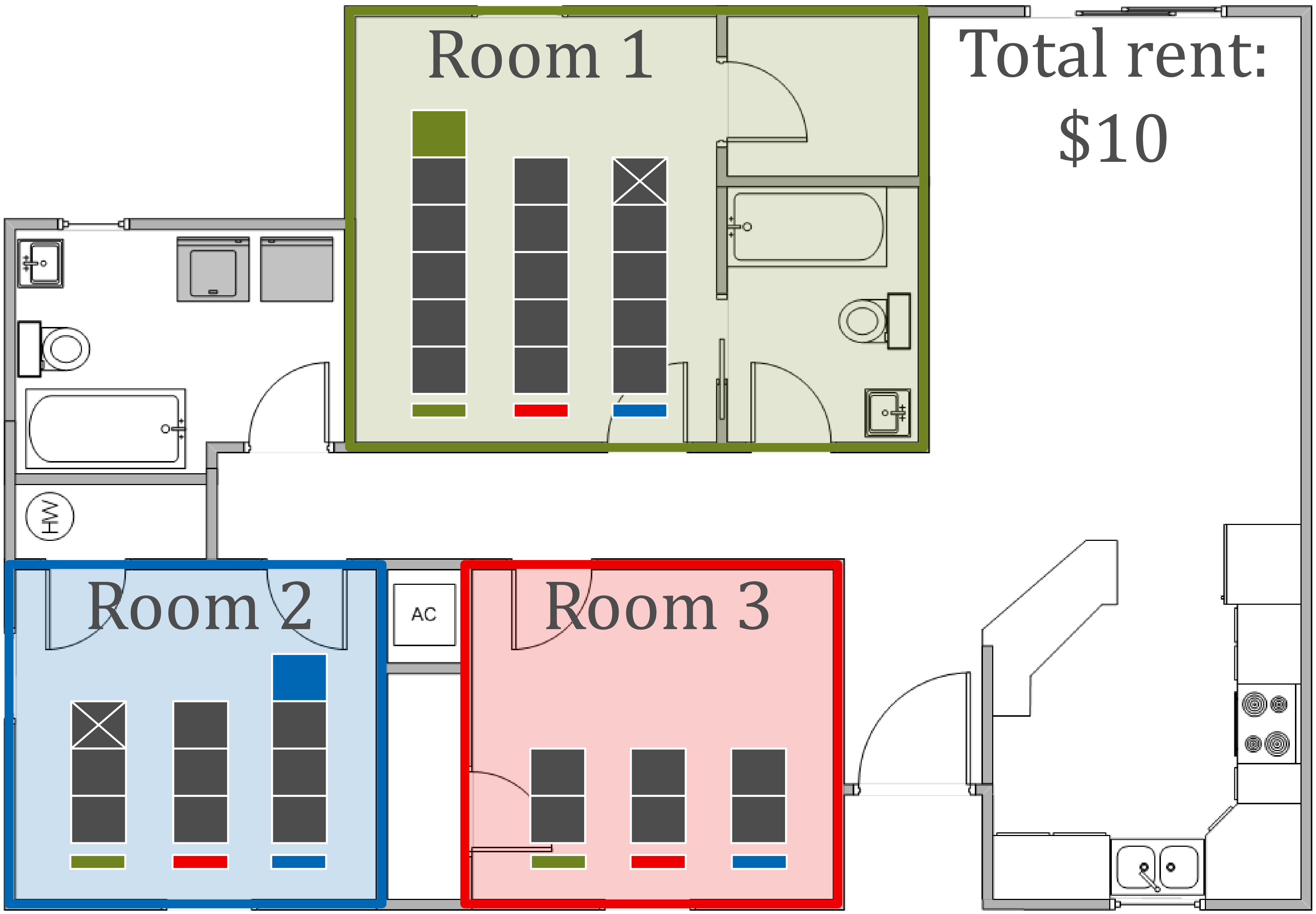
- Such a triangle is nothing but an approximately EF solution!
- By making the triangulation finer, we can approach envy-freeness
- Under additional closedness assumption, leads to existence of an EF solution ■

DISCUSSION

- It is possible to derive an algorithm from the proof
- Same techniques generalize to more housemates
- Same proof (with the original Sperner's Lemma) shows existence of EF cake division!

QUASI-LINEAR UTILITIES

- Suppose each player $i \in N$ has value v_{ir} for room r
- $\sum_r v_{ir} = R$, where R is the total rent
- The utility of player i for getting room r at price p_r is $v_{ir} - p_r$
- A solution consists of an assignment π and a price vector $\boldsymbol{p}$, where p_r is the price of room r
- Solution $(\pi, \boldsymbol{p})$ is envy free if
$$\forall i, j \in N, v_{i\pi(i)} - p_{\pi(i)} \geq v_{i\pi(j)} - p_{\pi(j)}$$
- Theorem [Svensson 1983]: An envy-free solution always exists under quasi-linearity


Room 1
Total rent:
$10
Room 2
AC
Room 3

PROPERTIES OF EF SOLUTIONS

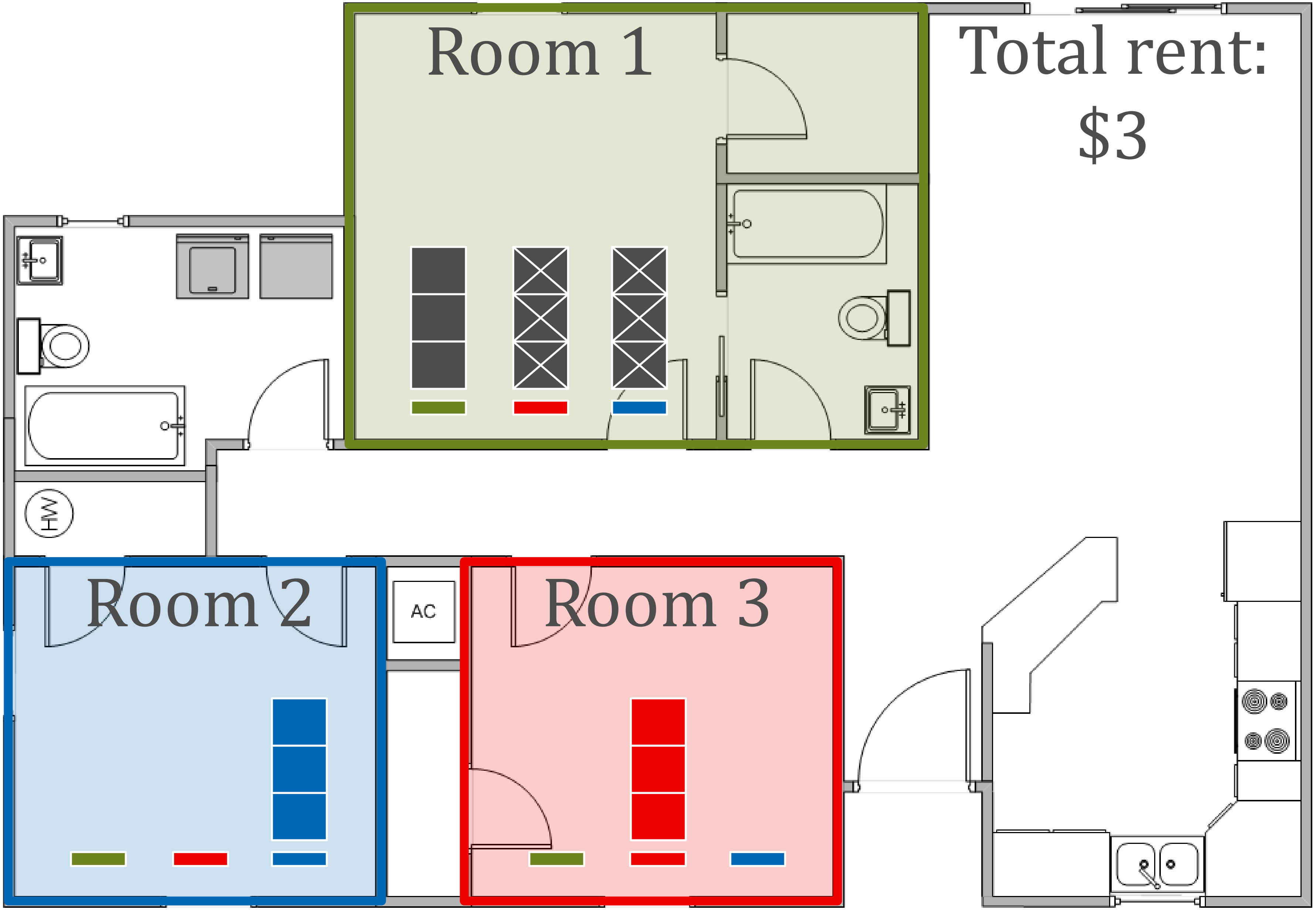
- Allocation π is welfare-maximizing if
$$\pi \in \text{argmax}_{\sigma} \sum_{i \in N} v_{i\sigma(i)}$$
- Lemma 1: If $(\pi, \boldsymbol{p})$ is an EF solution, then π is a welfare-maximizing assignment
- Lemma 2: If $(\pi, \boldsymbol{p})$ is an EF solution and σ is a welfare-maximizing assignment, then $(\sigma, \boldsymbol{p})$ is an EF solution, and for all i,
$$v_{i\pi(i)} - p_{\pi(i)} = v_{i\sigma(i)} - p_{\sigma(i)}$$

PROOF OF LEMMA 1

- Let $(\pi, \boldsymbol{p})$ be an EF solution, and let σ be another assignment
- Due to EF, for all i,
$$v_{i\pi(i)} - p_{\pi(i)} \geq v_{i\sigma(i)} - p_{\sigma(i)}$$
- Summing over all i,
$$\sum_{i \in N} v_{i\pi(i)} - \sum_{i \in N} p_{\pi(i)} \geq \sum_{i \in N} v_{i\sigma(i)} - \sum_{i \in N} p_{\sigma(i)}$$
- We get the desired inequality because prices sum up to R ■

POLYNOMIAL-TIME ALGORITHM

- Consider the algorithm that finds a welfare-maximizing assignment π, and then finds prices $\boldsymbol{p}$ that satisfy the EF constraint
- Theorem [Gal et al. 2017]: The algorithm always returns an EF solution, and can be implemented in polynomial time
- Proof:
  - We know that an EF solution $(\sigma, \boldsymbol{p})$ exists, by Lemma 2 $(\pi, \boldsymbol{p})$ is EF, so we would be able to find prices satisfying the EF constraint
  - The first part is max weight matching, the second part is a linear program ■


Room 1
Total rent:
$3
AC
HW
Room 2
Room 3

OPTIMAL EF SOLUTIONS

Straw Man Solution

Max sum of utilities
Subject to envy freeness

Maximin Solution

Max min utility
Subject to envy freeness

Equitable solution

Min max difference in utils
Subject to envy freeness

OPTIMAL EF SOLUTIONS

- Theorem [Gal et al. 2017]: The maximin and equitable solutions can be computed in polynomial time
- Theorem [Alkan et al. 1991]: The maximin solution is unique

Poll 1

Suppose that the values are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

What is the min utility under the maximin solution?

- $2/6 = 1/3$
- $2/8 = 1/4$
- $2/7$
- $2/9$

OPTIMAL EF SOLUTION

- Theorem [Gal et al. 2017]: The maximin solution is equitable, but not vice versa
- Rent division instance from Spliddit where the equitable solution is not maximin:

$$\begin{pmatrix} 2227 & 708 & 0 \\ 258 & 1378 & 1299 \\ 1000 & 1000 & 935 \end{pmatrix}$$

- Maximin solution gives room i to player i, with prices and utilities

$$\left(1813\frac{1}{3}, 600\frac{1}{3}, 521\frac{1}{3}\right), \left(413\frac{2}{3}, 777\frac{2}{3}, 413\frac{2}{3}\right)$$

- The max difference in utilities is 364
- The following prices and utilities have the same max difference, but lower minimum utility:

$$\left(1570\frac{2}{3}, 721\frac{2}{3}, 642\frac{2}{3}\right), \left(656\frac{1}{3}, 656\frac{1}{3}, 292\frac{1}{3}\right)$$

CAVEAT: STRATEGYPROOFNESS

- Lemma 1 tells us that any EF solution is welfare maximizing
- Therefore, any EF solution is Pareto efficient
- But there is no rent division algorithm that is both EF and Pareto efficient [Green and Laffont 1979]
- However, strategic behavior is largely a nonissue in practice in the rent division domain

CAVEAT: NEGATIVE RENT

- Envy-freeness may require negative rent, as the following example shows:

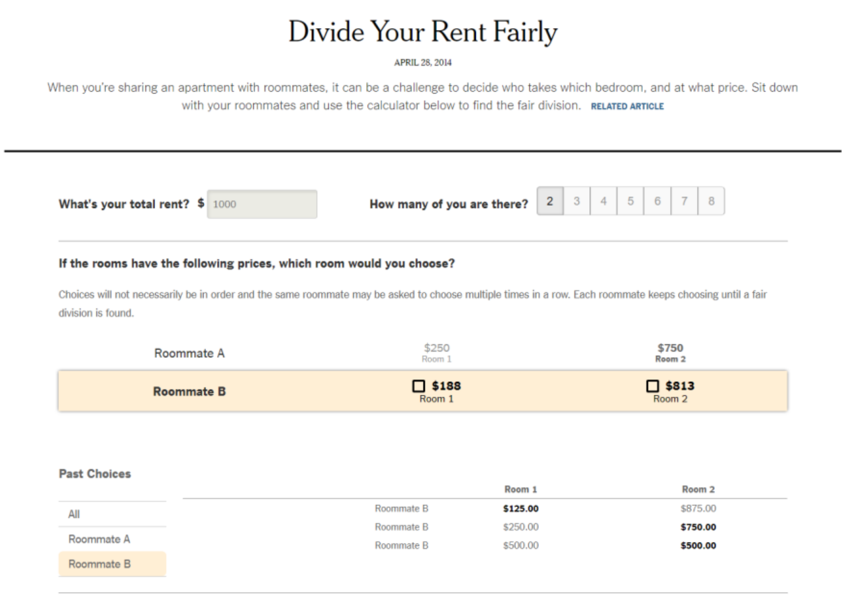
$$\begin{pmatrix} 36 & 34 & 30 & 0 \\ 31 & 36 & 33 & 0 \\ 34 & 30 & 36 & 0 \\ 32 & 33 & 35 & 0 \end{pmatrix}$$

- Whatever player i gets room 4 must pay 0, and the prices of the other rooms must be exactly his values to prevent envy
- Easy to verify that i can't be any of the players

WHICH MODEL IS BETTER?

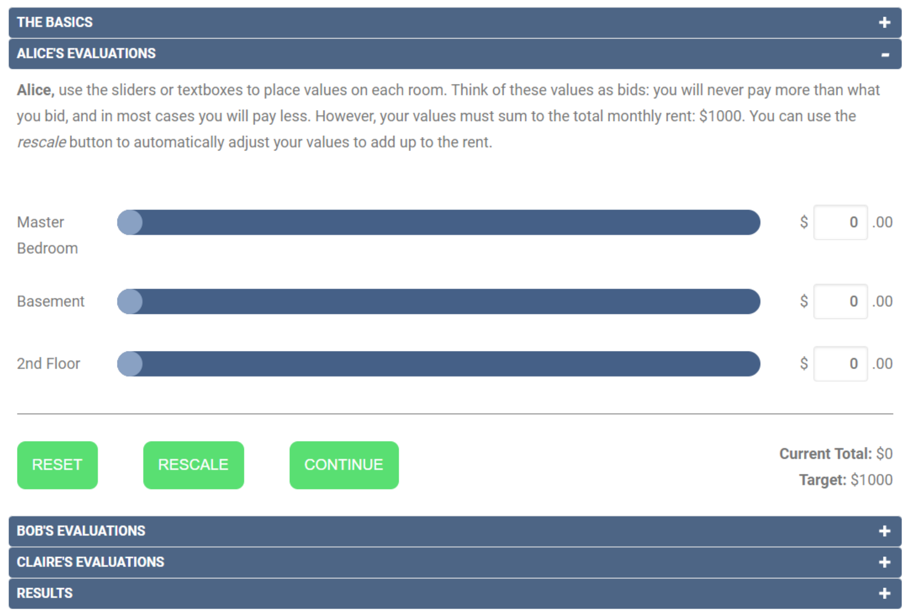
- Advantages of quasi-linear utilities:
  - Preference elicitation is easy: Each player reports a single number in one shot
  - Can choose among EF solutions
- Disadvantage of quasi-linear utilities: does not correctly model real-world situations
  - I want the room but I really can’t spend more than $500 on rent

INTERFACES



NY TIMES (rental harmony)

https://www.nytimes.com/interactive/2014/science/rent-division-calculator.html



Spliddit (quasi-linear utilities)

http://www.spliddit.org/apps/rent