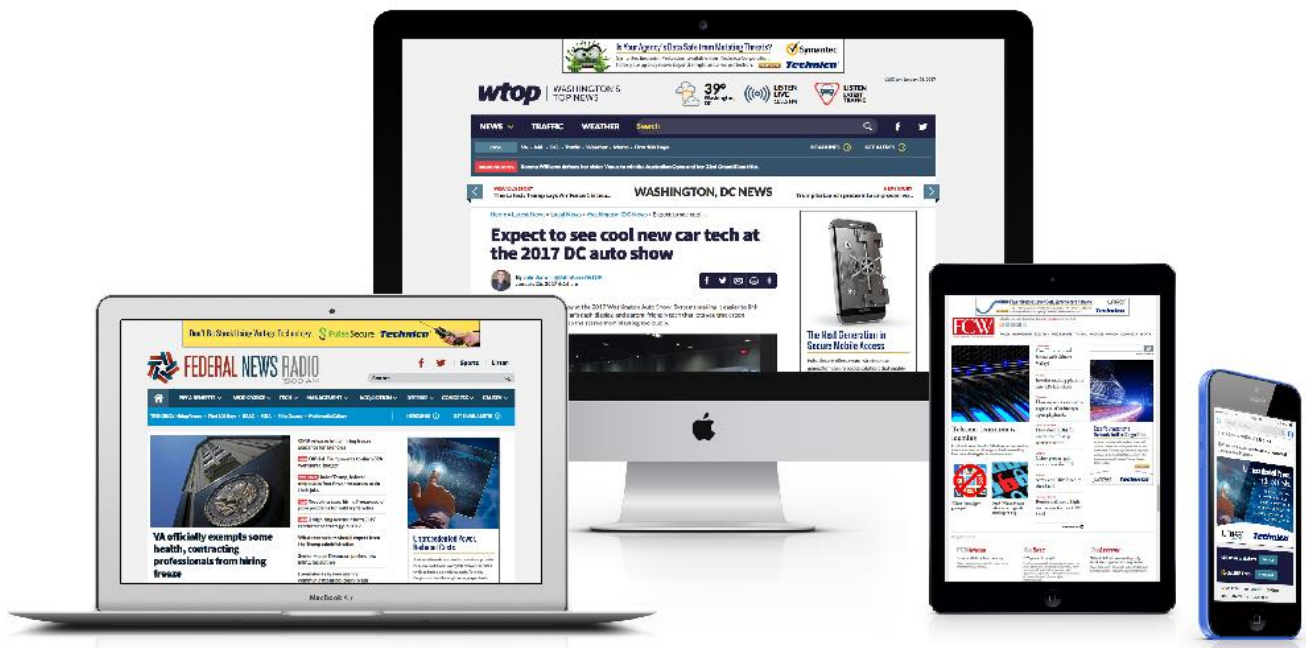
TRUTH JUSTICE ALGOS

Matching II: Online Algorithms

Teachers: Ariel Procaccia (this time) and Alex Psomas

DISPLAY ADVERTISING

- Display advertising is the largest matching problem in the world
- Bipartite graph with advertisers and impressions
- Advertisers specify which impressions are acceptable — this defines the edges
- Impressions arrive online



THE (SIMPLEST) MODEL

- Bipartite graph $G = (U, V, E)$ with $|U| = n$
- U is known "offline," the vertices of V arrive online (with their incident edges)
- Online vertices can only be matched when they arrive
- Objective: maximize size of matching
- ALG has competitive ratio $\alpha \leq 1$ if for every graph G and every input order π of V,

$$\frac{ALG(G,\pi)}{OPT(G)} \geq \alpha$$

ALGORITHM GREEDY

- **Algorithm GREEDY:** match to an arbitrary unmatched neighbor (if one exists)

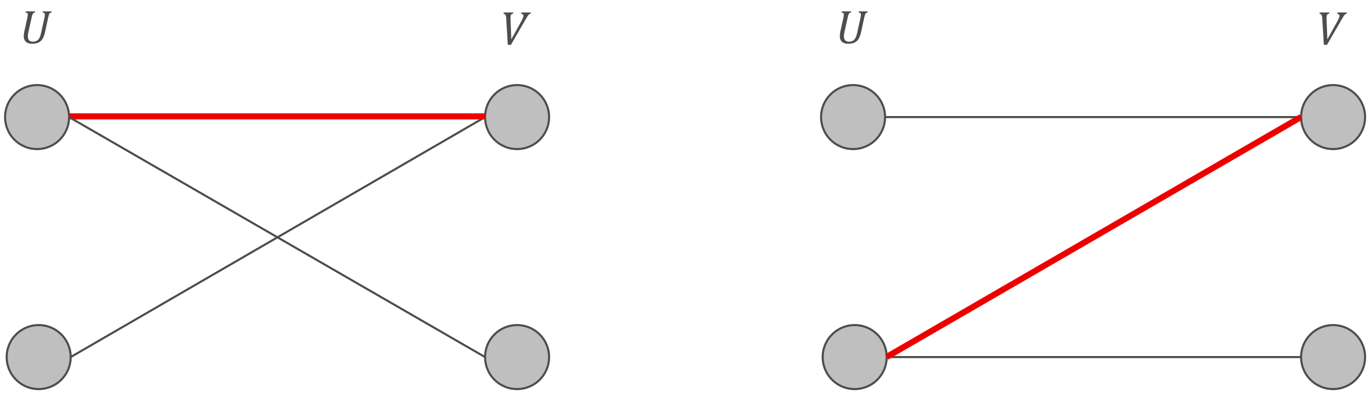
Poll 1

Competitive ratio of GREEDY?

- $1/n$
- $1/\log n$
- $1/\sqrt{n}$
- $1/2$

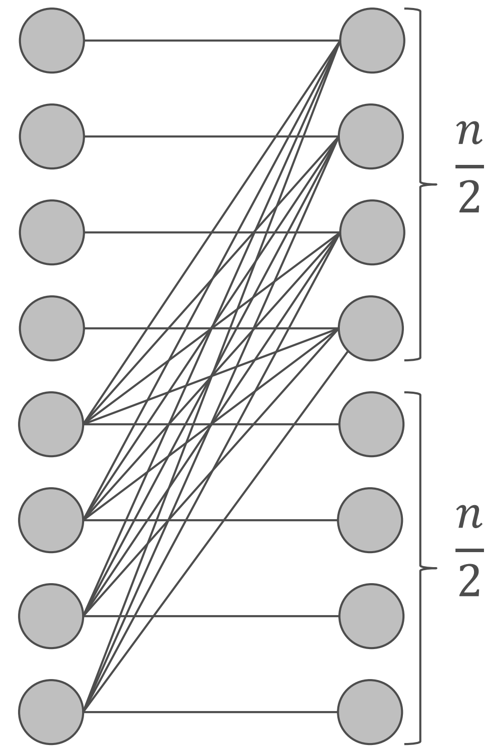
UPPER BOUND

- **Observation:** The competitive ratio of any deterministic algorithm is at most 1/2



TAKE 2: ALGORITHM RANDOM

- Obvious idea: randomness
- Algorithm RANDOM: Match to an unmatched neighbor uniformly at random
- Achieves ¾ on previous example



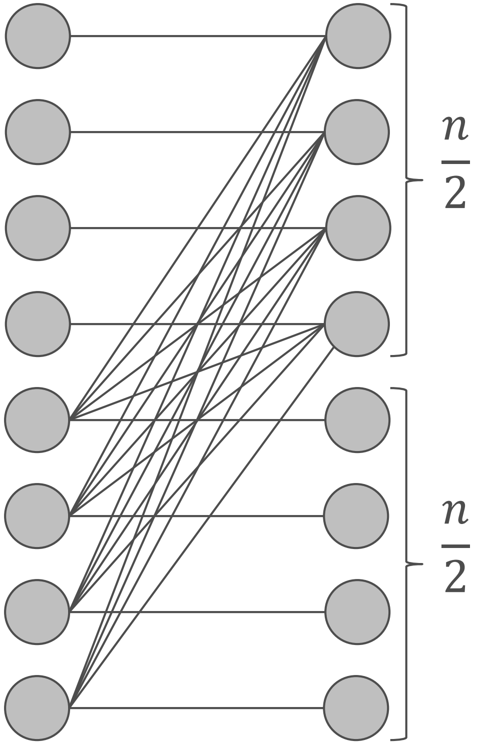
Poll 2

Competitive ratio of RANDOM on current graph?

- ~7/8
- ~6/8
- ~5/8
- ~4/8

TAKE 3: ALGORITHM RANKING

- Algorithm RANKING:
  - Choose a random permutation $\pi: U \to [n]$
  - Match each vertex to its unmatched neighbor u with the lowest $\pi(u)$
- Looks like this is doing better than RANDOM on previous example!
- Theorem [Karp et al. 1990]: The competitive ratio of RANKING is $1 - 1/e \approx 0.63$

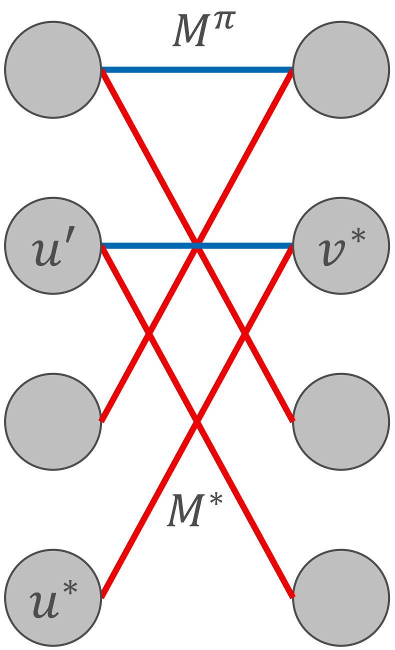


PROOF OF THEOREM

- Assume for ease of exposition that $\mathrm{OPT} = n$
- Fix a perfect matching $M^*: U \cup V \to U \cup V$
- Fix π and $u \in U$
- If u is matched under π, (π, u) is a match event at position $\pi(u)$, otherwise miss event
- ALG is the sum of probabilities of match events at all positions

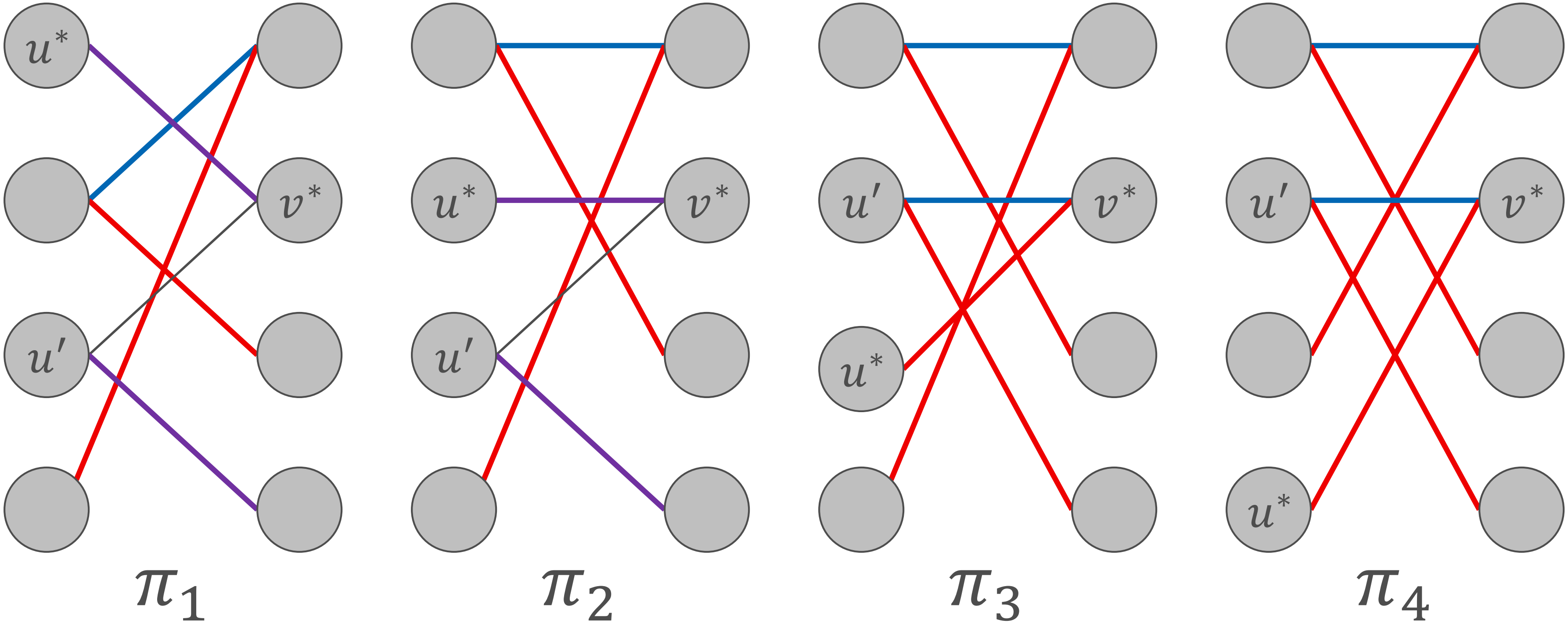
PROOF OF THEOREM

- π induces a matching M^{π}
- Consider a miss event (π, u^*) with $\pi(u^*) = t$
- $v^* = M^*(u^*), u' = M^{\pi}(v^*)$
- Define π_i by moving u^* to position $i = 1, \ldots, \mathrm{n}$
- Claim: for each i, $M^{\pi_i}(v^*) = u_i$ with $\pi_i(u_i) \leq t$



PROOF OF THEOREM

- Proof of claim: by illustration



PROOF OF THEOREM

- We have a 1-n mapping between miss events (π, u^*) and match events (π_i, u_i) where $M^{\pi_i}(u_i) = M^*(u^*)$ and $\pi_i(u_i) \leq \pi(u^*)$
- Claim: Each miss event at position t is mapped to n unique match events
- Proof of claim:
  - Fix miss events (π, u) and (π', u') such that $\pi(u) = \pi'(u') = t$, and both are mapped to $(\hat{\pi}, \hat{u})$
  - $M^{\hat{\pi}}(\hat{u}) = M^*(u)$ and $M^{\hat{\pi}}(\hat{u}) = M^*(u') \Rightarrow u = u'$
  - The map only moves u from position t in π and π', giving $\hat{\pi}$ in both cases $\Rightarrow \pi = \pi'$ ■

PROOF OF THEOREM

- We get the following set of equations for every $t = 1, \ldots, n$:

$$n \cdot \Pr[\text{Miss at } t] \leq \sum_{s \leq t} \Pr[\text{Match at } s]$$

- Setting $x_t = \Pr[\text{Match at } t]$, this is

$$1 - x_t \leq \frac{1}{n} \sum_{s \leq t} x_s$$

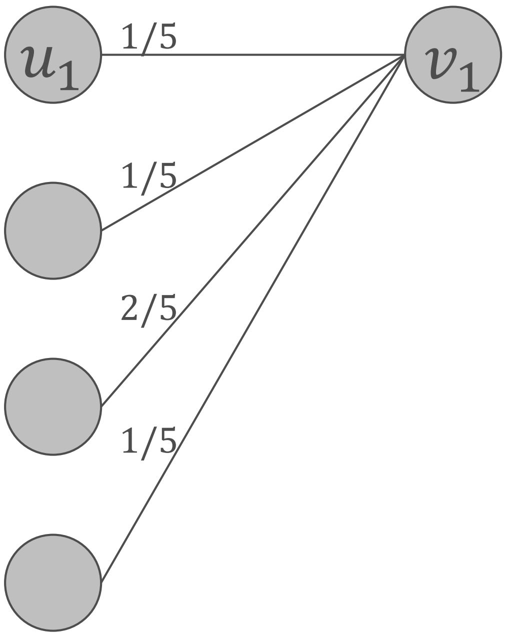
- By minimizing the objective function $\sum_t x_t$ over this polytope, we get $\sum_t x_t \geq \left(1 - \frac{1}{e}\right) n$ ■

UPPER BOUND

- Theorem [Karp et al. 1990]: No randomized alg has competitive ratio better than
$$1 - 1/e + o(1)$$
- The proof below is due to Wajc [2015]
- Fractional algorithm: deterministically assign fractional weights to edges such that s.t. $\forall u \in U \cup V, f(u) = \sum_{(u,v)\in E} w_{uv} \leq 1$
- Lemma [Wajc 2015]: For any randomized alg there is a fractional alg with at least the same competitive ratio

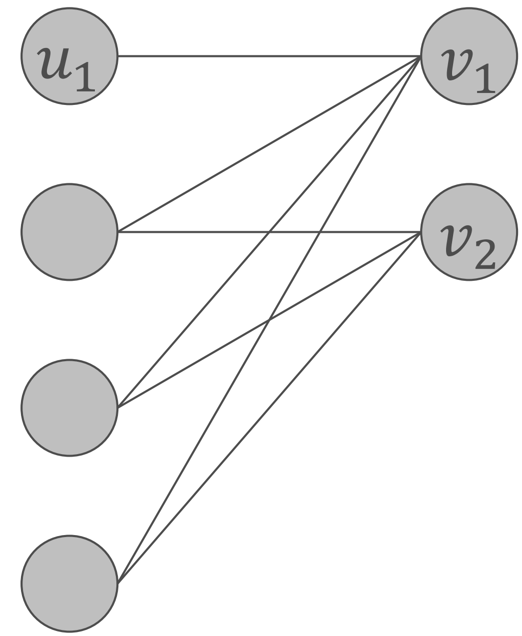
PROOF OF THEOREM

- First online vertex v_1 is connected to all U
- Let $u_1 \in \operatorname{argmin}_{u \in U} f(u)$, in particular $f(u_1) \leq 1/n$
- u_1 will not be connected to any future online vertex



PROOF OF THEOREM

- t-th online vertex v_t is connected to all $U \backslash \{u_1, \dots, u_{t-1}\}$
- $u_t \in \mathrm{argmin}_{u \in U \backslash \{u_1, \dots, u_{t-1}\}} f(u)$
- u_t will not be connected to any future online vertex



Poll 3

What is OPT?

- $n/2$
- $n\left(1 - \frac{1}{e}\right)$
- $3n/4$
- n

PROOF OF THEOREM

- After step t, offline vertices that continue to be matched are matched to an average of at least $f(u) = \sum_{k=1}^{t} \frac{1}{n-k+1}$
- Following the arrival of the t-th online vertex with $t = n\left(1 - \frac{1}{e}\right) + 1$, it holds that offline vertices that will neighbor future online vertices are matched to an average of

$$f(u) = \sum_{k=1}^{n\left(1-\frac{1}{e}\right)+1} \frac{1}{n-k+1} = \sum_{k=\frac{n}{e}}^{n} \frac{1}{k} \geq \ln(n) - \ln\frac{n}{e} = 1$$

PROOF OF THEOREM

- So at step t, $\frac{1}{n-t}\sum_{k=t+1}^{n} f(u_k) \geq 1$, but because $f(u) \leq 1$ for all $u \in U$, this means that $f(u_k) = 1$ for all
$$k = t+1, \dots, n$$
- That is, the algorithm cannot match the vertices $v_{t+1}, \dots, v_n$
- $\text{ALG} \leq n\left(1 - \frac{1}{e}\right) + 1$ ■