TRUTH JUSTICE ALGOS

Fair Division II:
Envy-Free Cake Cutting

Teachers: Ariel Procaccia and Alex Psomas (this time)

STORY SO FAR

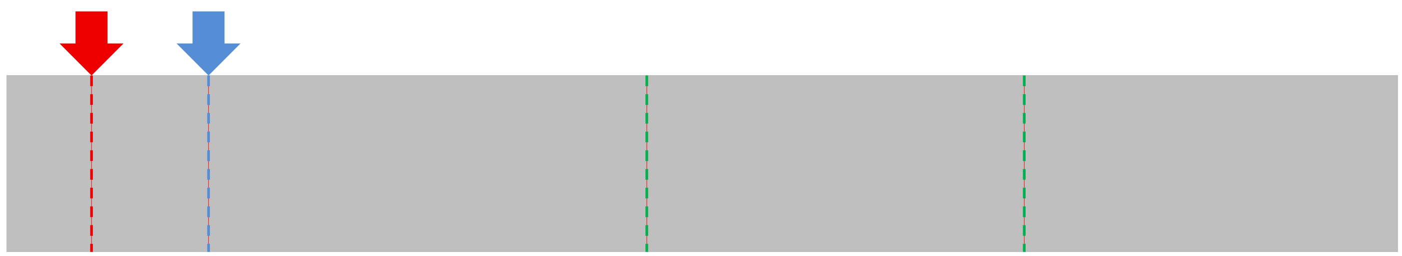
- (Bounded) EF protocol for 3 players by Selfridge and Conway (1960).
- (Bounded) EF protocol for 4 players by Aziz and Mackenzie (2016).
  - 584 or so queries.
  - Improved to 181 by ACFM**P**V 2018.

Today

- (Bounded) EF protocol for n players by Aziz and Mackenzie (2016).
  - $O(n^{n^{n^{n^{n^{n}}}}})$ queries.
- Lower bound $\Omega(n^2)$ [Procaccia 2009]

A DIFFERENT ENVY FREE PROTOCOL FOR 3 PLAYERS

1. Player A cuts in 3 equal (to her) pieces.
2. If B and C have different favorite pieces we're done.
   a) Otherwise, B and C have **competition** for their favorite item.
3. Players B and C make a **2-mark**: make the favorite piece equal to the second favorite piece.



B: #1 #2 #3

C: #1 #3 #2

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4. Give the marked piece to the agent with the rightmost mark (B in this case), starting from the leftmost mark. The other agent (C in this case) picks her favorite piece. The cutter (A in this case) takes the last piece.

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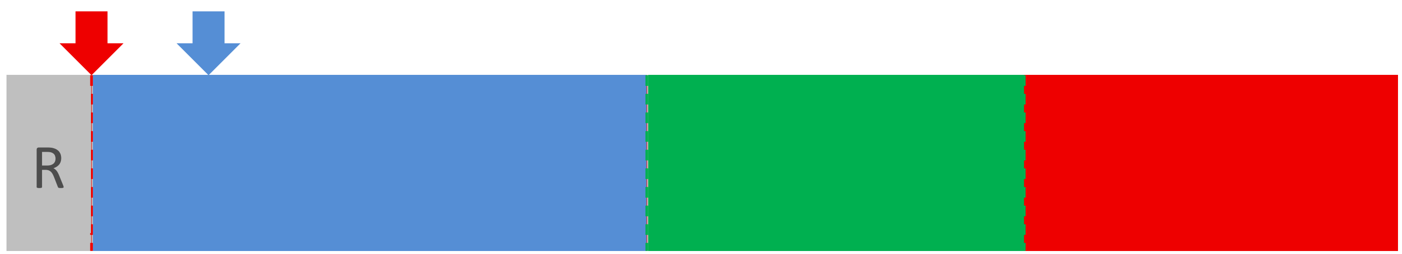
A DIFFERENT ENVY FREE PROTOCOL FOR 3 PLAYERS

1. The allocation is partial (R is not allocated), but envy free.
2. The cutter (A in this case) and one more agent (C) got whole pieces, i.e. value of at least 2/3 according to A has been allocated, and A got value 1/3.
3. The cutter (A) would not envy player B, even if B got all of the residue R.
   a) Definition: In a partial allocation, player i **dominates** player j, if i would not envy j even if j got all of the residue.



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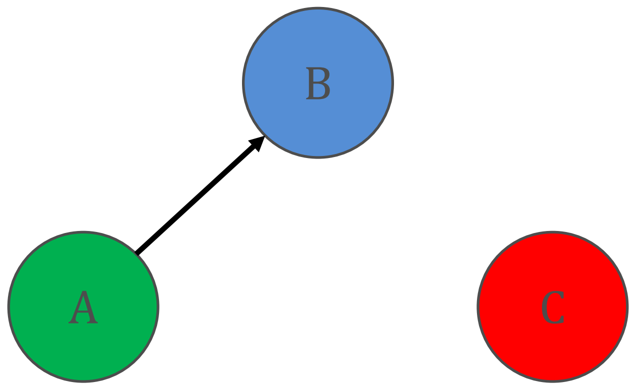
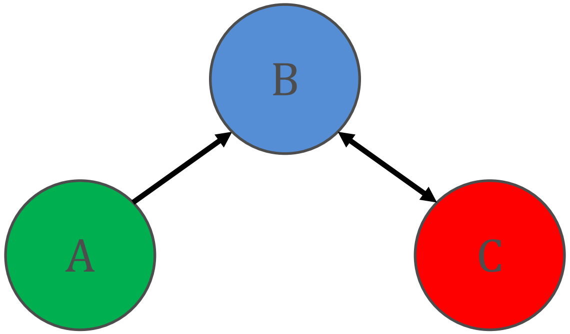
A DIFFERENT ENVY FREE PROTOCOL FOR 3 PLAYERS

- We call the protocol we used **CORE.**
- Domination graph:

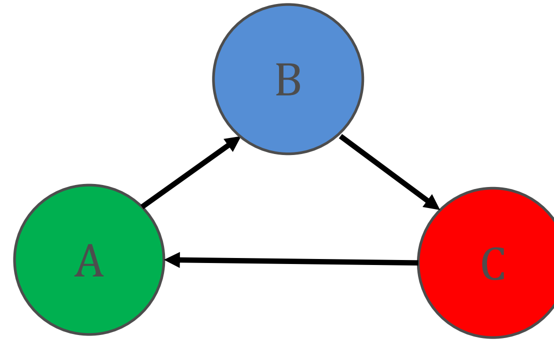


- Idea: execute **CORE** on the residue!
- Who should be the cutter?
  - Observation: If A is the cutter, we are not guaranteed a new edge.
  - If (say) C is the cutter, we get a new edge from C to another player.
  - Idea: Run **CORE** with every agent as the cutter?

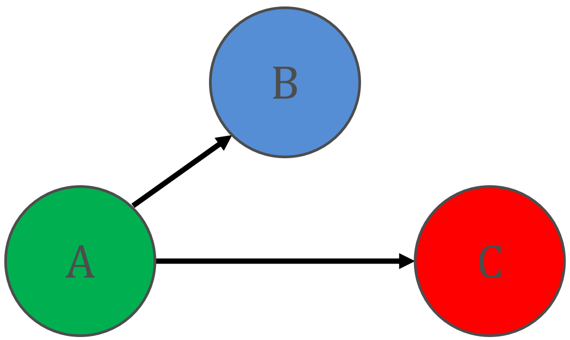
A DIFFERENT ENVY FREE PROTOCOL FOR 3 PLAYERS



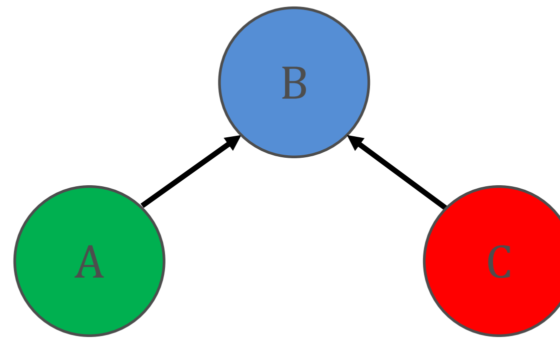
Graph 1



Graph 2



Graph 3



Graph 4

Which domination graph is non-trivial?

1. Graph 1
2. Graph 2
3. Graph 3
4. Graph 4

A DIFFERENT ENVY FREE PROTOCOL FOR 3 PLAYERS

- We'll aim for Graph 3.
- Execute **CORE** two more times, with A as the cutter.
- Worst case scenario: B gets the "marked" piece every time!
- $A^k = \{ p_A^k, p_B^k, p_C^k \}$ is the allocation in the k-th execution of CORE.
- **Key definition:** The **gain** of a player i in an allocation A^k is the difference between $v_i(p_i^k)$ and the maximum value of i for a piece in A^k given to an agent i does not dominate.
- $G_{A^k}(i) = v_i(p_i^k) - \max_{j \neq \mathrm{i}} v_i(p_j^k)$, and j is not dominated by i.

A DIFFERENT ENVY FREE PROTOCOL FOR 3 PLAYERS

	A^1	A^2	A^3
Player A	$G_{A^1}(A) = 0$	$G_{A^2}(A) = 0$	$G_{A^3}(A) = 0$
Player B	$G_{A^1}(B)$	$G_{A^2}(B)$	$G_{A^3}(B)$
Player C	$G_{A^1}(C)$	$G_{A^2}(C)$	$G_{A^2}(C)$

- Claim 1: For some k, $G_{A^k}(i) \leq \sum_{\ell \neq k} G_{A^\ell}(i)$, for all i.
- Proof:
  - Player A is not an issue.
  - For every $i \in \{B, C\}$, only $argmax_\ell G_{A^\ell}(i)$ could be a problem.
  - Therefore, every $i \in \{B, C\}$ excludes (at most) one column.
  - The remaining allocation/column works.

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Player B	$G_{A^1}(B)$	$G_{A^2}(B)$	$G_{A^3}(B)$
Player C	$G_{A^1}(C)$	$G_{A^2}(C)$	$G_{A^2}(C)$

- Claim 1: For some k, $G_{A^k}(i) \leq \sum_{\ell \neq k} G_{A^\ell}(i)$, for all i.
- Say A^2 is the allocation guaranteed to exist by Claim 1.
- Make the following **correction**: C gets B's piece. B picks a new piece (i.e. her second favorite piece). A gets the last piece. Let $\widehat{A^2}$ be the new allocation.
  - Observation: A and B get a "whole" piece (wrt the original cuts of A).
- Claim 2: $G_{\widehat{A^2}}(i) \geq -G_{A^2}(i)$ for all players i.
  - Trivial for A.
  - The gain of C is zero in both allocations.
  - $G_{A^2}(B)$ is the value of B for the piece between the two marks. $G_{\widehat{A^2}}(B)$ is minus that.

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- Claim 1: For some k, $G_{A^k}(i) \leq \sum_{\ell \neq k} G_{A^\ell}(i)$, for all i.
- Claim 2: $G_{\widehat{A^2}}(i) \geq -G_{A^2}(i)$ for all players i.
- Claims $1 + 2$ imply that the overall allocation, $A^1 \cup \widehat{A^2} \cup A^3$, is partial envy free.
- Since C got the marked piece in $\widehat{A^2}$, then A dominates C.
- Therefore, A dominates both B and C.
- Cut-and-choose between B and C.

RECAP

- **CORE**:
  - **Cutter** cuts in equal pieces.
  - If there is **competition**, non-cutters make a 2-mark to their favorite piece. Player with the rightmost mark gets the marked piece, starting from the 2^{nd} rightmost mark.
- Run **CORE** with the same cutter 3 times. Hope for the cutter to dominate two different players.
  - If this doesn't happen naturally, find an allocation where **gain** is very small for everyone.
  - Make a **correction** to this allocation, in a way that (1) the overall allocation remains partial envy free, (2) cutter dominates a new player.

EF PROTOCOL FOR 4 PLAYERS: PHASE 1

- **CORE**:
  - **Cutter** cuts in equal pieces.
  - If there is **competition**, non-cutters make a 2-mark to their favorite piece. Player with the rightmost mark gets the marked piece, starting from the 2nd rightmost mark.
- Run **CORE** with the same cutter 3 times. Hope for the cutter to dominate two different players.
  - If this doesn't happen naturally, find an allocation where **gain** is very small for everyone.
  - Make a **correction** to this allocation, in a way that (1) the overall allocation remains partial envy free, (2) cutter dominates a new player.

EF PROTOCOL FOR 4 PLAYERS

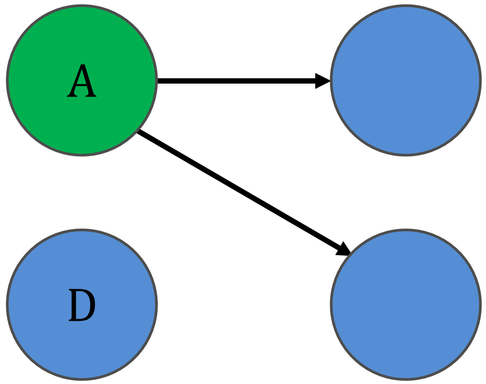
- More complicated definition of competition
  - We have to take into account the 2^{nd} favorite piece (out of 4).
- There are 2-marks, making the favorite equal to the 2^{nd} favorite, or 3-marks, making the first two equal to the 3^{rd} favorite.
- **CORE** satisfies the following properties:
  - The cutter and at least one more player receive whole pieces.
  - The allocation is partial envy-free.
- **CORE** is executed 4 times instead of 3.
  - Gain is defined the same way. Same argument goes through.
- **CORRECTION** is a separate protocol. Takes as input an allocation of **CORE**. Properties:
  - The insignificant piece changes hands.
  - If a non-cutter was allocated her favorite unmarked piece, she gets the same value.
  - $G_{\hat{A}}(i) \geq -G_A(i)$ for all players i.

EF PROTOCOL FOR 4 PLAYERS

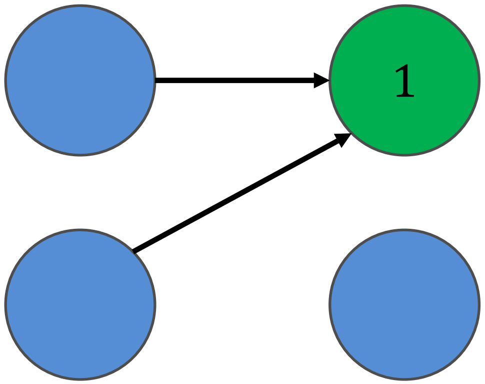
- Claim: Executing **CORE** with the same player as cutter twice results in a domination edge. More specifically, the cutter dominates the player who in the first execution received the **insignificant** piece, the piece that the cutter values the least.
- Proof:
  - Say exactly 2 pieces were partially allocated. If exactly 1 piece was partially allocated the proof is easier, and it can't be more than 2 partial pieces.
  - Piece p_A goes to the cutter A, and p_1 is the insignificant piece.
  - $R = r_1 \cup r_2$, where $r_1$is the unallocated part of the insignificant piece, i.e. $v_A(r_1) + v_A(p_1) = v_A(p_A) = 1/4$.
  - $v_A(r_1) \geq v_A(r_2)$, therefore $v_A(R) \leq 2v_A(r_1)$.
  - Let R' be the new residue (possibly between executions of CORE with a different cutter). Then $v_A(R') \leq v_A(R)$.
  - Let R'' be the residue after the 2nd execution with A as the cutter.
  - $v_A(R'') \leq \frac{2}{4} v_A(R')$, since 2 of the 4 pieces where allocated fully.
  - So, $v_A(R'') \leq v_A(r_1) = v_A(p_A) - v_A(p_1)$.

PHASE 1

- Domination graph so far:



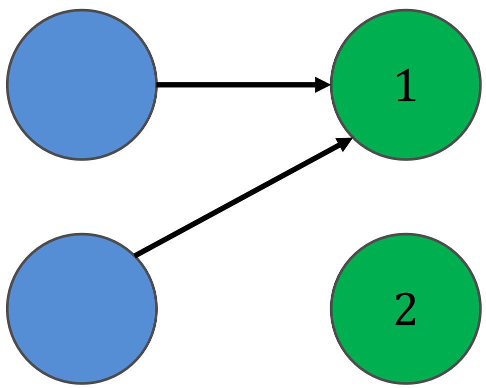
- Target graph: some player is dominated by two other players. (extra edges are omitted)



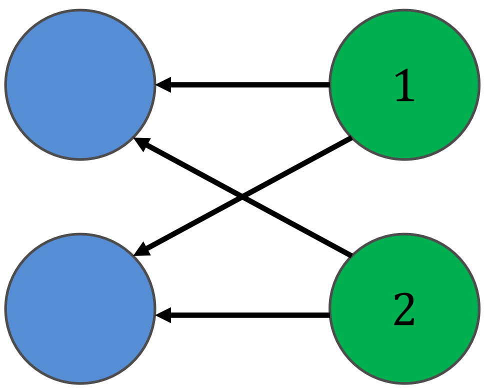
- It suffices to execute **CORE** once, with D as the cutter.
  - Main idea: A does not “compete” with anyone. Then, 3 full pieces are allocated.

PHASE 2

- Domination graph so far:



- Target graph (extra edges omitted):



- Execute **CORE** twice, with 2 as the cutter, and **CORRECTION** if necessary.
- **CORE** property 3: Assume we run **CORE** with 2 as the cutter, and suppose agent 1 is dominated by the other two non-cutters, 3 and 4, neither of whom dominates the other. Then, (1) 1 gets her favorite of the four complete pieces without making any marks, (2) at least three complete pieces are allocated, and (3) if a non-cutter, say 3, gets a partial piece, then the remaining non-cutter, 4, is indifferent between her piece and 3 ’s piece.

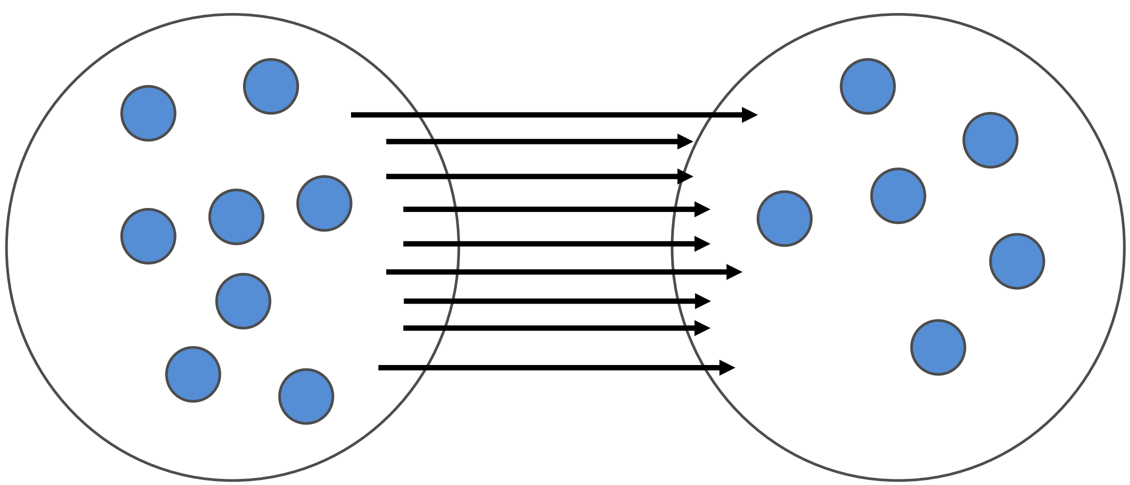
RECAP

- Phase 1: Get to a graph where 3 and 4 dominate 1.
- Phase 2: Get to a graph where 1 and 2 dominate 3 and 4.
- Phase 3: Cut-and-choose with 3 and 4.

BEYOND 4 PLAYERS

- Again, **CORE** does all of work.
- Very similar properties.
  - Partial envy-free.
  - A cutter cuts into n equal pieces. **CORE** allocates two whole pieces, one of them to the cutter.
- Very very complicated "corrections".
- Target dominance graph:

Everyone in the left set dominates everyone in the right set



BEYOND 4 PLAYERS: TOWARDS A SIMPLER ALGORITHM

- Very simple **CORE** [DFHY 18']:
- Player cuts in n equal pieces.
- There exists an ordering where the i-th player receives the i-th piece, trimmed so that there is no envy with the first i-1 players.
  - First player gets first piece.
  - Second player trims second piece (if necessary) to make it equal to the first.
  - And so on.
- The allocation is partial envy free.
- Try every ordering of players and pieces.