TRUTH JUSTICE ALGOS

Matching I:
Kidney Exchange

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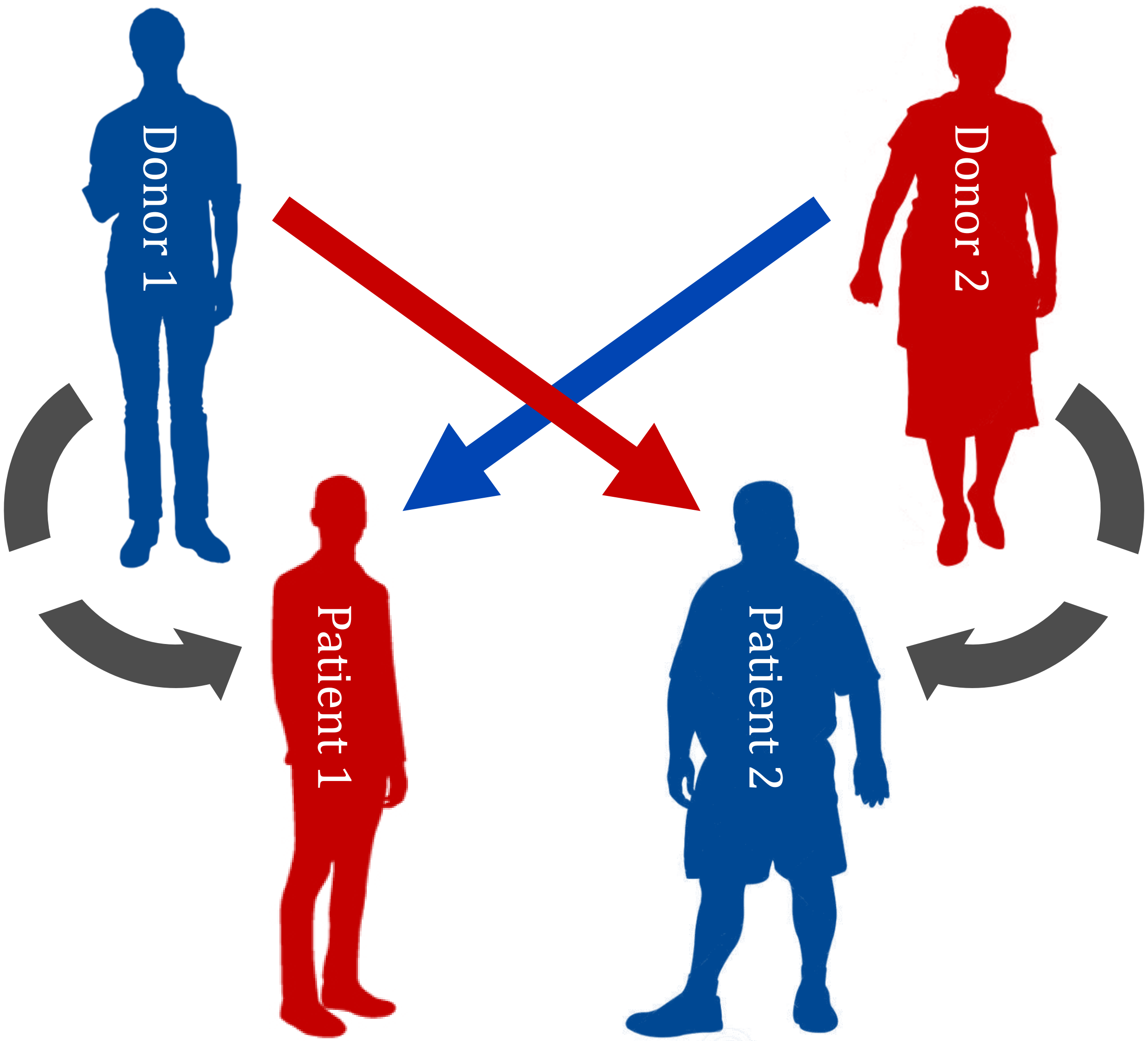
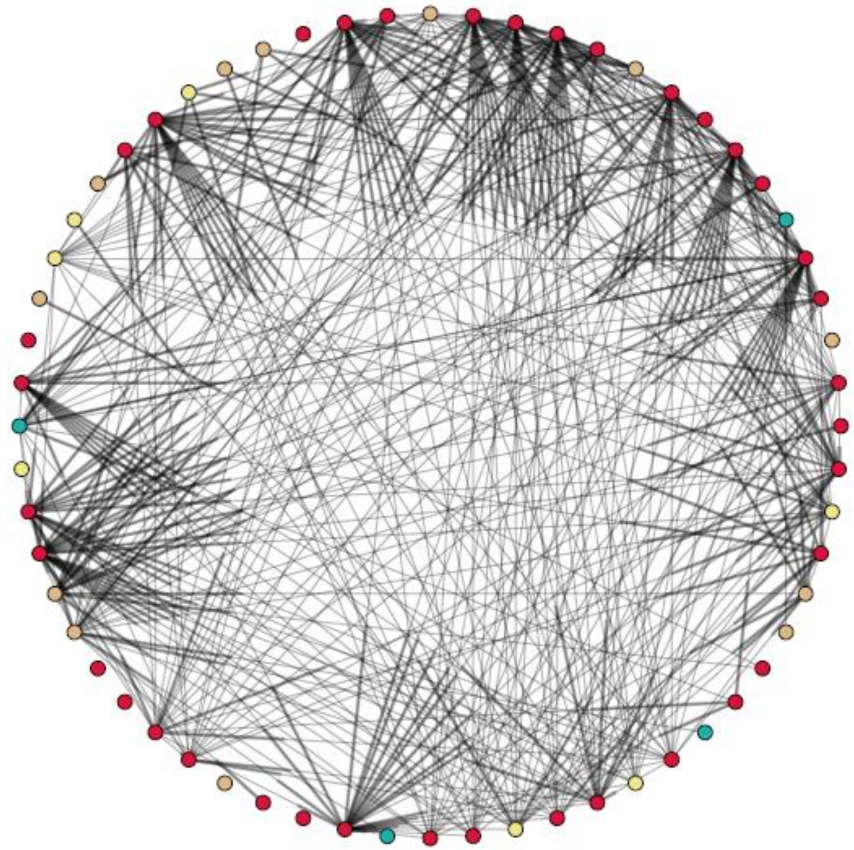
KIDNEY EXCHANGE



EXAMPLE: KIDNEY EXCHANGE

- CYCLE-COVER: Given a directed graph G and $L \in \mathbb{N}$, find a collection of disjoint cycles of length $\leq L$ in G that maximizes the number of covered vertices
- The problem is:
  - Easy for $L = 2$ (why?)
  - Easy for unbounded L
  - NP-hard for a constant $L \geq 3$



UNOS pool, Dec 2010
[Courtesy John Dickerson]

APPLICATION: UNOS

UNOS

DONATE
LIFE

UNITED NETWORK FOR ORGAN SHARING

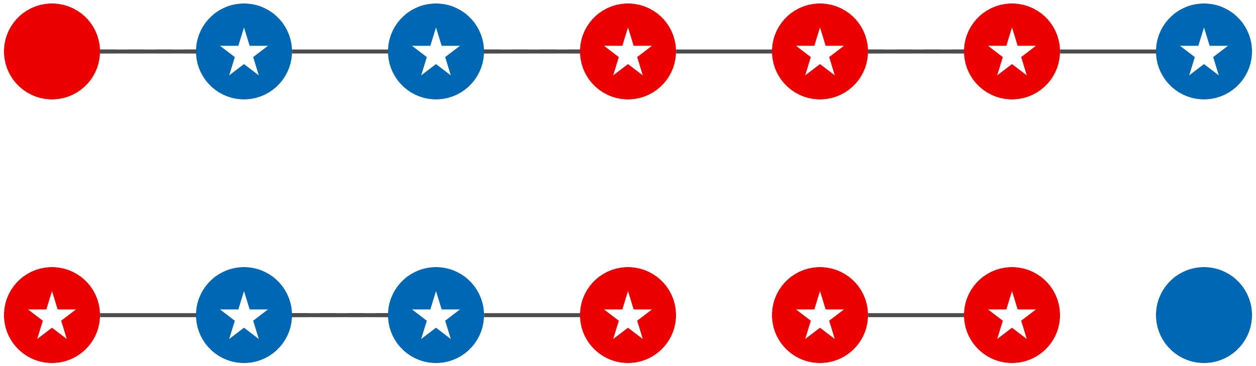
INCENTIVES

- In the past kidney exchanges were carried out by individual hospitals
- Today there are nationally organized exchanges; participating hospitals have little other interaction
- It was observed that hospitals match easy-to-match pairs internally, and enroll only hard-to-match pairs into larger exchanges
- Goal: incentivize hospitals to enroll all their pairs

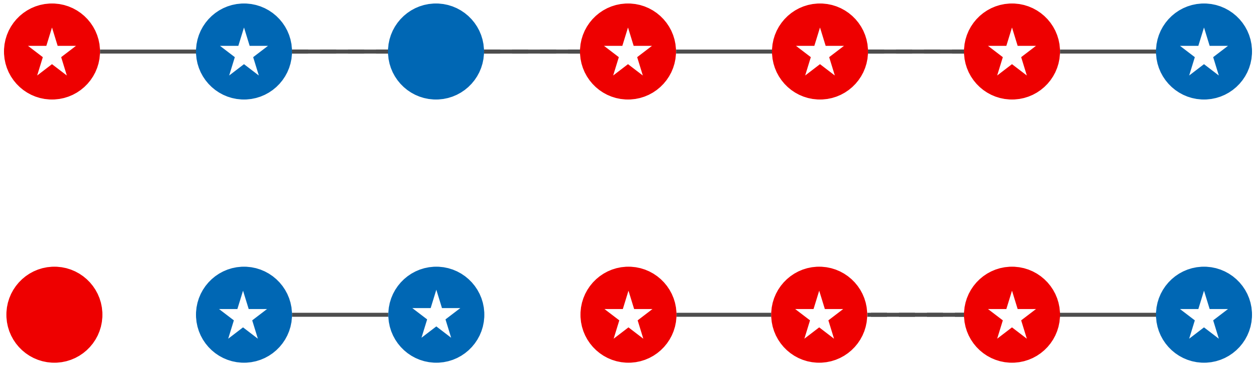
THE STRATEGIC MODEL

- Undirected graph (only pairwise matches!)
  - Vertices = donor-patient pairs
  - Edges = compatibility
  - Each player controls subset of vertices
- Mechanism receives a graph and returns a matching
- Utility of player = # its matched vertices
- Target: # matched vertices (util. social welfare)
- Strategy: subset of revealed vertices
  - But edges are public knowledge
- Mechanism is strategyproof (SP) if it is a dominant strategy to reveal all vertices

OPT IS MANIPULABLE



OPT IS MANIPULABLE



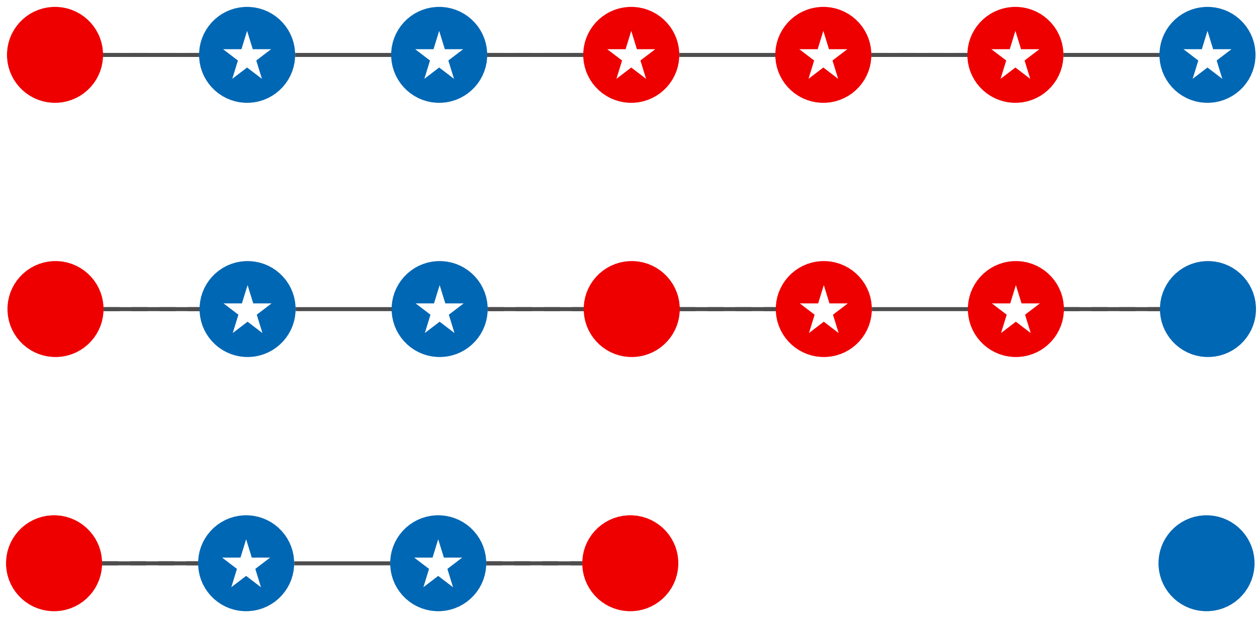
APPROXIMATING SW

- Theorem [Ashlagi et al. 2010]: No deterministic SP mechanism can give a $2 - \epsilon$ approximation
- Proof: We just proved it!
- Theorem [Kroer and Kurokawa 2013]: No randomized SP mechanism can give a $\frac{6}{5} - \epsilon$ approximation
- Proof: Homework 4

SP MECHANISM: TAKE 1

- Assume two players
- The MATCH$_{\{\{1\},\{2\}\}}$ mechanism:
  - Consider matchings that maximize the number of "internal edges"
  - Among these return a matching with max cardinality

ANOTHER EXAMPLE



GUARANTEES

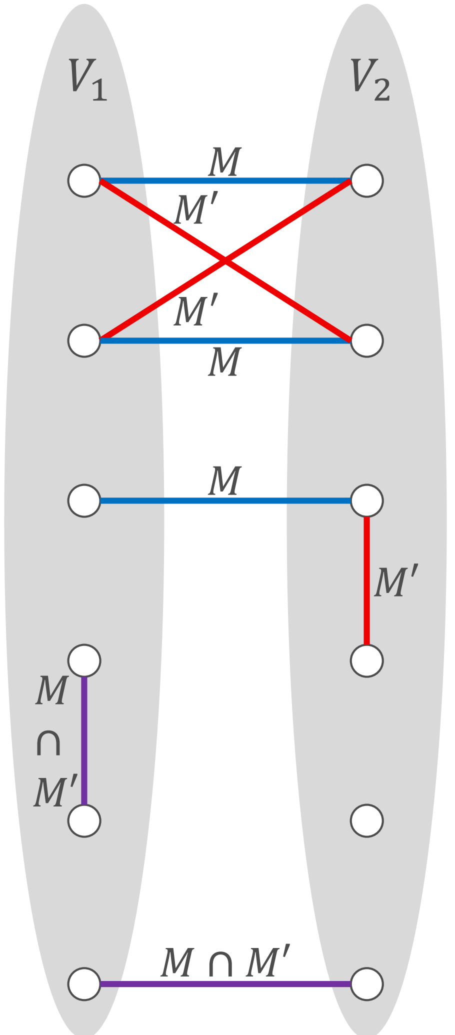
- $\textsc{Match}_{\{\{1\},\{2\}\}}$ gives a 2-approximation
  - Cannot add more edges to matching
  - For each edge in optimal matching, one of the two vertices is in mechanism's matching
- Theorem (special case): $\textsc{Match}_{\{\{1\},\{2\}\}}$ is strategyproof for two players

PROOF OF THEOREM

- M = matching when player 1 is honest, M' = matching when player 1 hides vertices
- $M \Delta M'$ consists of paths and even-length cycles, each consisting of alternating M, M' edges

Question

What's wrong with the illustration on the right?

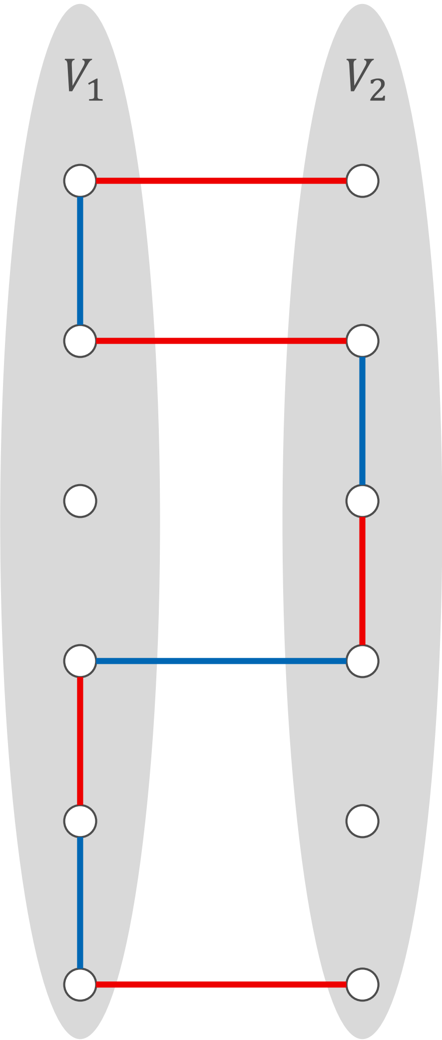


PROOF OF THEOREM

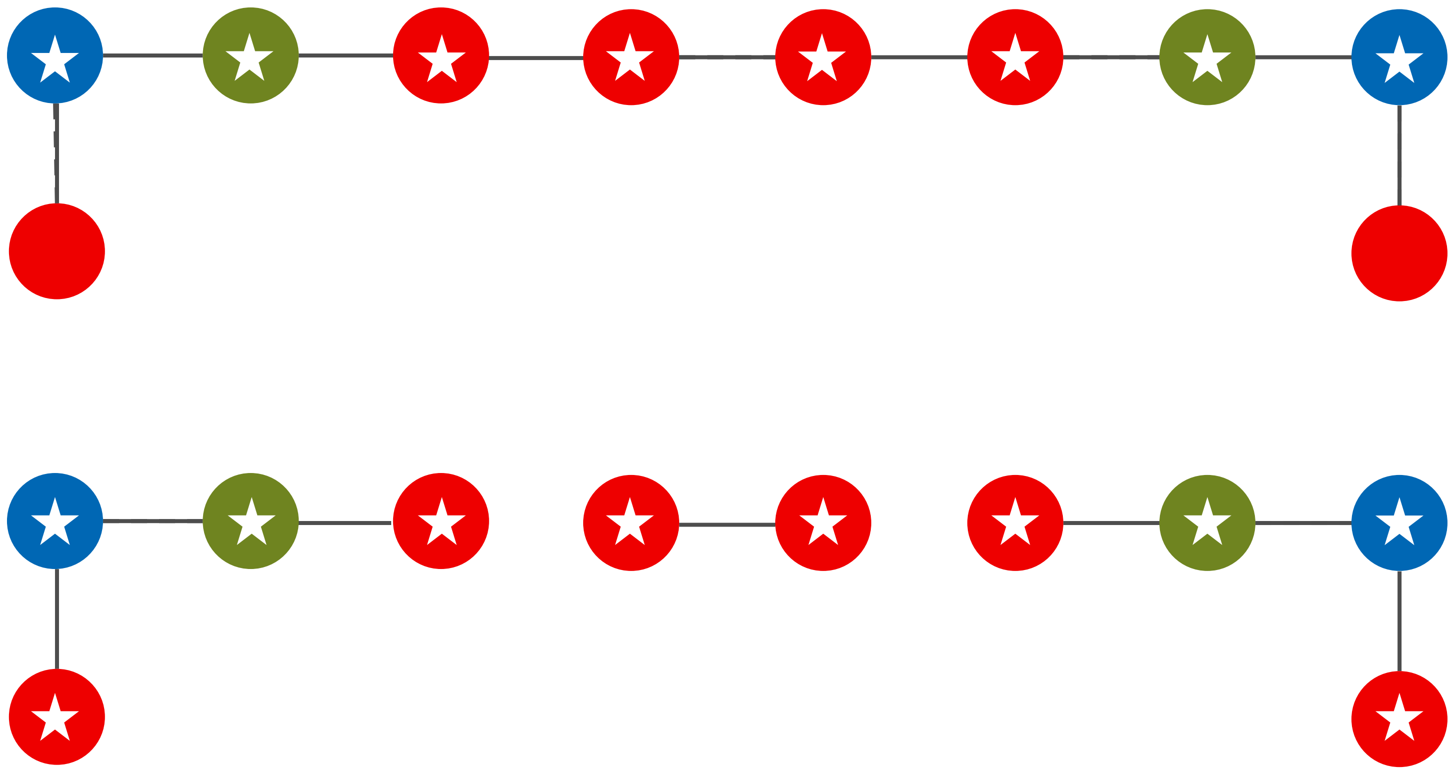
- Consider a path in $M \Delta M'$, denote its edges in M by P and its edges in M' by P'
- For $i, j \in \{1,2\}$,
  $$P_{ij} = \{(u, v) \in P : u \in V_i, v \in V_j\}$$
  $$P'_{ij} = \{(u, v) \in P' : u \in V_i, v \in V_j\}$$
- $|P_{11}| \geq |P'_{11}|$, suppose $|P_{11}| = |P'_{11}|$
- It holds that $|P_{22}| = |P'_{22}|$
- M is max cardinality $\Rightarrow |P_{12}| \geq |P'_{12}|$
- $U_1(P) = 2|P_{11}| + |P_{12}| \geq 2|P'_{11}| + |P'_{12}| = U_1(P')$

PROOF OF THEOREM

- Suppose $|P_{11}| > |P'_{11}|$
- $|P_{12}| \geq |P'_{12}| - 2$
  - Every subpath within V_2 is of even length
  - We can pair the edges of P_{12} and P'_{12}, except maybe the first and the last
- $U_1(P) = 2|P_{11}| + |P_{12}| \geq 2(|P'_{11}| + 1) + |P'_{12}| - 2 = U_1(P')$ ■



THE CASE OF 3 PLAYERS



SP MECHANISM: TAKE 2

- Let $\Pi = (\Pi_1, \Pi_2)$ be a bipartition of the players
- The $\textsc{Match}_\Pi$ mechanism:
  - Consider matchings that maximize the number of "internal edges" and do not have any edges between different players on the same side of the partition
  - Among these return a matching with max cardinality (need tie breaking)

EUREKA?

- Theorem [Ashlagi et al. 2010]: $\textsc{Match}_{\Pi}$ is strategyproof for any number of players and any partition Π
- Recall: for $n = 2$, $\textsc{Match}_{\{\{1\},\{2\}\}}$ guarantees a 2-approximation

Poll 1

Approximation guarantees given by $\textsc{Match}_{\Pi}$ for $n = 3$ and $\Pi = \{\{1\},\{2,3\}\}$?

- 2-approx
- 3-approx
- 4-approx
- More than 4

THE MECHANISM

- The Mix-and-Match mechanism:
  - Mix: choose a random partition Π
  - Match: Execute Match_{Π}
- Theorem [Ashlagi et al. 2010]: Mix-and-Match is strategyproof and guarantees a 2-approximation
- We only prove the approximation ratio

PROOF OF THEOREM

- $M^* =$ optimal matching
- Create a matching M' such that M' is max cardinality on each V_i, and

$$\sum_i |M'_{ii}| + \frac{1}{2}\sum_{i \neq j} |M'_{ij}| \geq \sum_i |M^*_{ii}| + \frac{1}{2}\sum_{i \neq j} |M^*_{ij}|$$

  - $M^{**} =$ max cardinality on each V_i
  - For each path P in $M^* \Delta M^{**}$, add $P \cap M^{**}$ to M' if M^{**} has more internal edges than M^*, otherwise add $P \cap M^*$ to M'
  - For every internal edge M' gains relative to M^*, it loses at most one edge overall ■

PROOF OF THEOREM

- Fix Π and let M^{Π} be the output of $\textsc{Match}_{\Pi}$
- The mechanism returns max cardinality across Π subject to being max cardinality internally, therefore

$$\sum_{i} |M_{ii}^{\Pi}| + \sum_{i \in \Pi_1, j \in \Pi_2} |M_{ij}^{\Pi}| \geq \sum_{i} |M'_{ii}| + \sum_{i \in \Pi_1, j \in \Pi_2} |M'_{ij}|$$

PROOF OF THEOREM

$$\mathbb{E}\left[\left|M^{\Pi}\right|\right] = \frac{1}{2^n}\sum_{\Pi}\left(\sum_{i}\left|M_{ii}^{\Pi}\right| + \sum_{i\in\Pi_1, j\in\Pi_2}\left|M_{ij}^{\Pi}\right|\right)$$

$$\geq \frac{1}{2^n}\sum_{\Pi}\left(\sum_{i}\left|M_{ii}'\right| + \sum_{i\in\Pi_1, j\in\Pi_2}\left|M_{ij}'\right|\right)$$

$$= \sum_{i}\left|M_{ii}'\right| + \frac{1}{2^n}\sum_{\Pi}\sum_{i\in\Pi_1, j\in\Pi_2}\left|M_{ij}'\right|$$

$$= \sum_{i}\left|M_{ii}'\right| + \frac{1}{2}\sum_{i\neq j}\left|M_{ij}'\right| \geq \sum_{i}\left|M_{ii}^{*}\right| + \frac{1}{2}\sum_{i\neq j}\left|M_{ij}^{*}\right|$$

$$\geq \frac{1}{2}\sum_{i}\left|M_{ii}^{*}\right| + \frac{1}{2}\sum_{i\neq j}\left|M_{ij}^{*}\right| = \frac{1}{2}\left|M^{*}\right| \quad \blacksquare$$