Truth Justice Algos

Social Choice IV: Restricted Preferences

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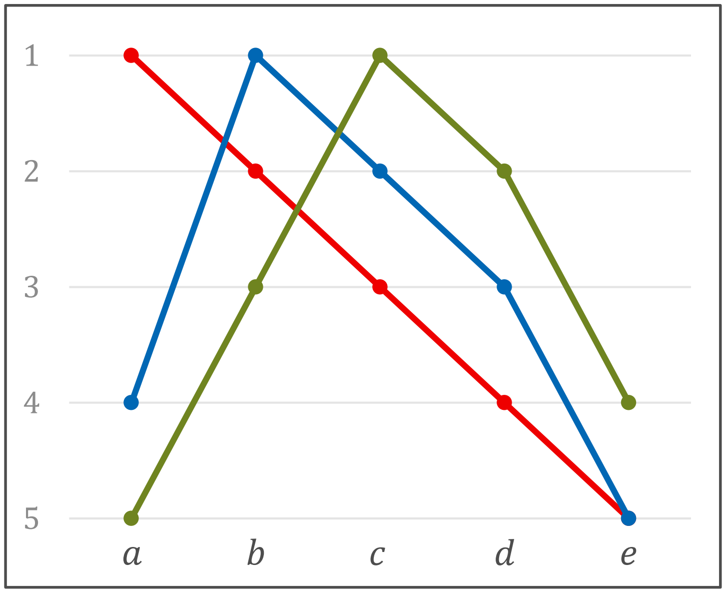
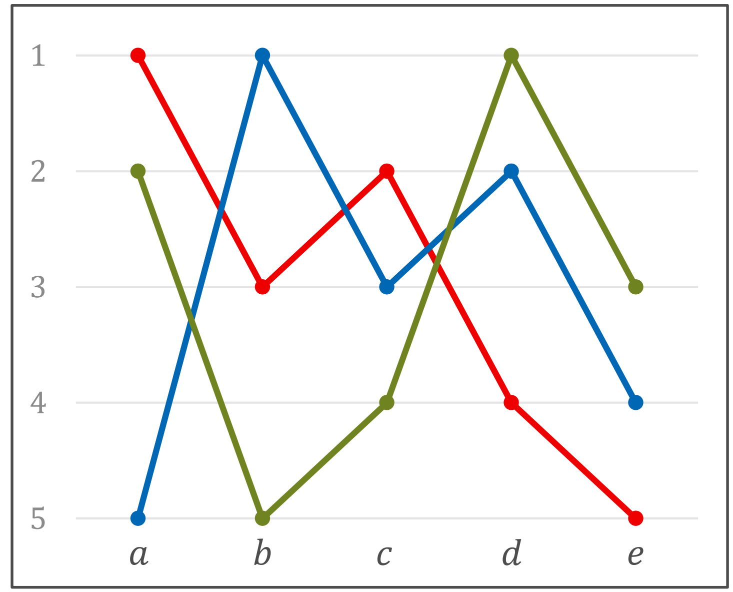
SINGLE-PEAKED PREFERENCES

- The Gibbard-Satterthwaite Theorem requires a full preference domain, i.e., each ranking of the alternatives is possible
- Can we circumvent the theorem if we restrict the preferences in reasonable ways?
- Assume an ordering $\leq$ over the set of alternatives A
- Voter i has single-peaked preferences if there is a peak $x^* \in A$ such that $y < z \leq x^* \Rightarrow z \succ_i y$ and $y > z \geq x^* \Rightarrow z \succ_i y$

SINGLE-PEAKED PREFERENCES

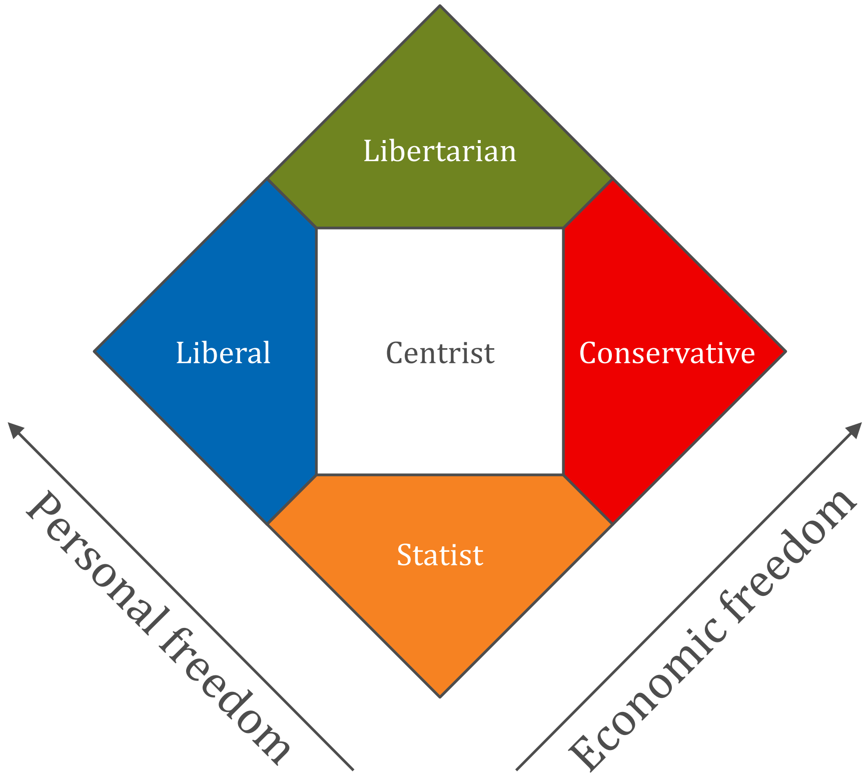


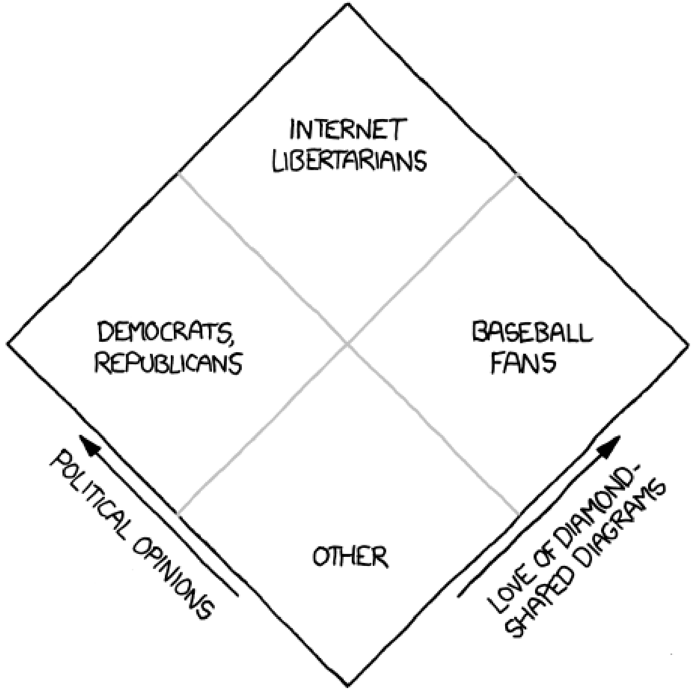
Single peaked



Not single peaked

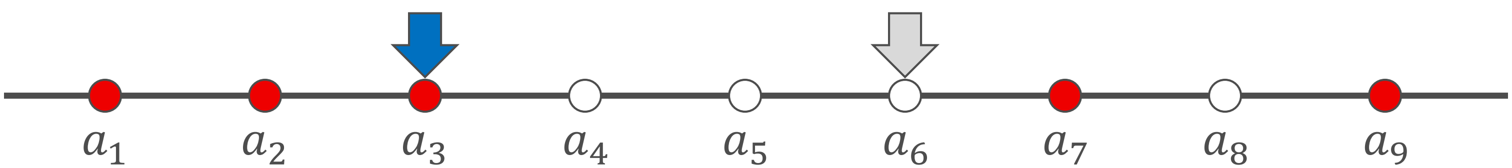
EXAMPLE: NOLAN CHART



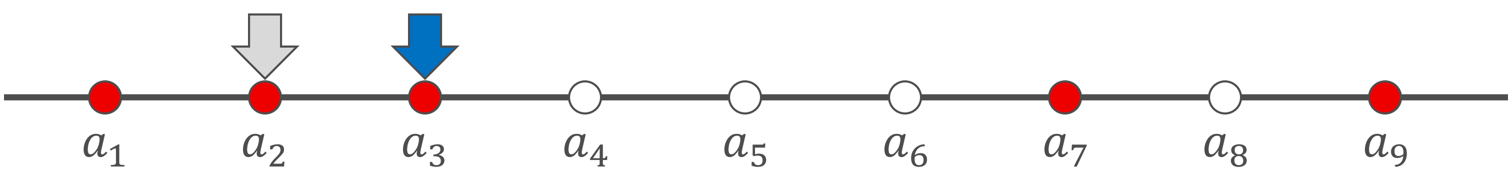


SINGLE-PEAKED PREFERENCES

- Assume an odd number of voters with single-peaked preferences, then a Condorcet winner exists, and is given by the median peak



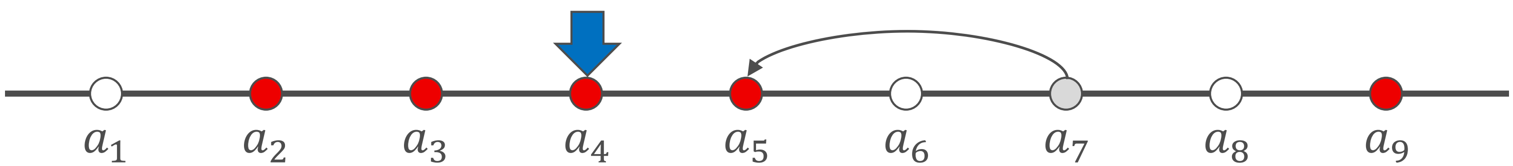
A majority of voters prefer the median to any alternative to its right



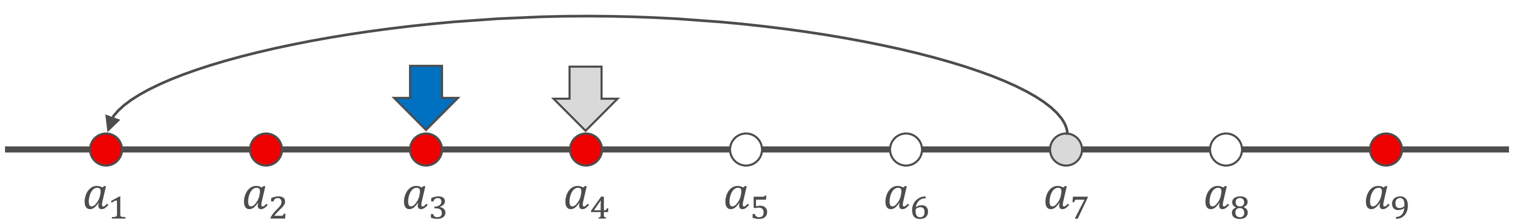
A majority of voters prefer the median to any alternative to its left

STRATEGYPROOF RULES

- Assume voters with single-peaked preferences, then the voting rule that selects the median peak is strategyproof



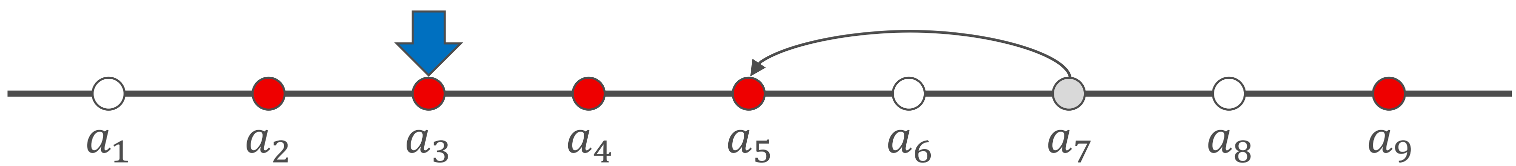
Reporting another peak on the same side of the median makes no difference



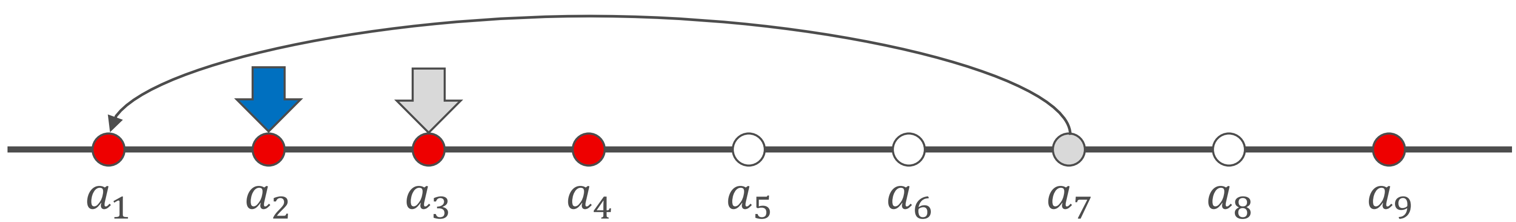
Reporting another peak on the other side of the median make things worse

STRATEGYPROOF RULES

- Assume voters with single-peaked preferences, then the voting rule that selects the kth order statistic is strategyproof



Reporting another peak on the same side of the 2nd order static makes no difference



Reporting another peak on the other side of the 2nd order statistic make things worse

STRATEGYPROOF RULES

- For single-peaked preferences σ_i, denote the peak by $P(\sigma_i)$
- Theorem [Moulin 1980]: An anonymous voting on single-peaked preferences is SP iff there exist $p_1, \ldots, p_{n+1} \in A$ (called phantoms) such that, for every profile $\boldsymbol{\sigma}$,
$$f(\boldsymbol{\sigma}) = \mathrm{med}\big(p_1, \ldots, p_n, P(\sigma_1), \ldots, P(\sigma_n)\big)$$
- Examples:
  - Median (odd n): $(n+1)/2$ phantoms at each of a_1 and a_m
  - Second order statistic: $n-1$ phantoms at a_1, two at a_m
  - $f \equiv x$ (constant function): $n+1$ phantoms at x

FACILITY LOCATION

- Each player $i \in N$ has a location $x_i \in \mathbb{R}$
- Given $\boldsymbol{x} = (x_1, \dots, x_n)$, choose a facility location $f(\boldsymbol{x}) = y \in \mathbb{R}$
- $\text{cost}(y, x_i) = |y - x_i|$
- This defines (very specific) single-peaked preferences over the set of alternatives $\mathbb{R}$, where the peak of player i is x_i

FACILITY LOCATION

- Two objective functions
  - Social cost: $\mathrm{sc}(y, \boldsymbol{x}) = \sum_i |y - x_i|$
  - Maximum cost: $\mathrm{mc}(y, \boldsymbol{x}) = \max_i |y - x_i|$
- For the social choice objective, the median is optimal and SP
- For the maximum cost objective, the optimal solution is $(\min x_i + \max x_i)/2$, but it is not SP

DETERMINISTIC RULES FOR MC

- We say that a deterministic rule f gives an α-approximation to the max cost if for all $\boldsymbol{x} \in \mathbb{R}^n$,

$$\mathrm{mc}(f(\boldsymbol{x}), \boldsymbol{x}) \leq \alpha \cdot \min_{y \in \mathbb{R}} \mathrm{mc}(y, \boldsymbol{x})$$

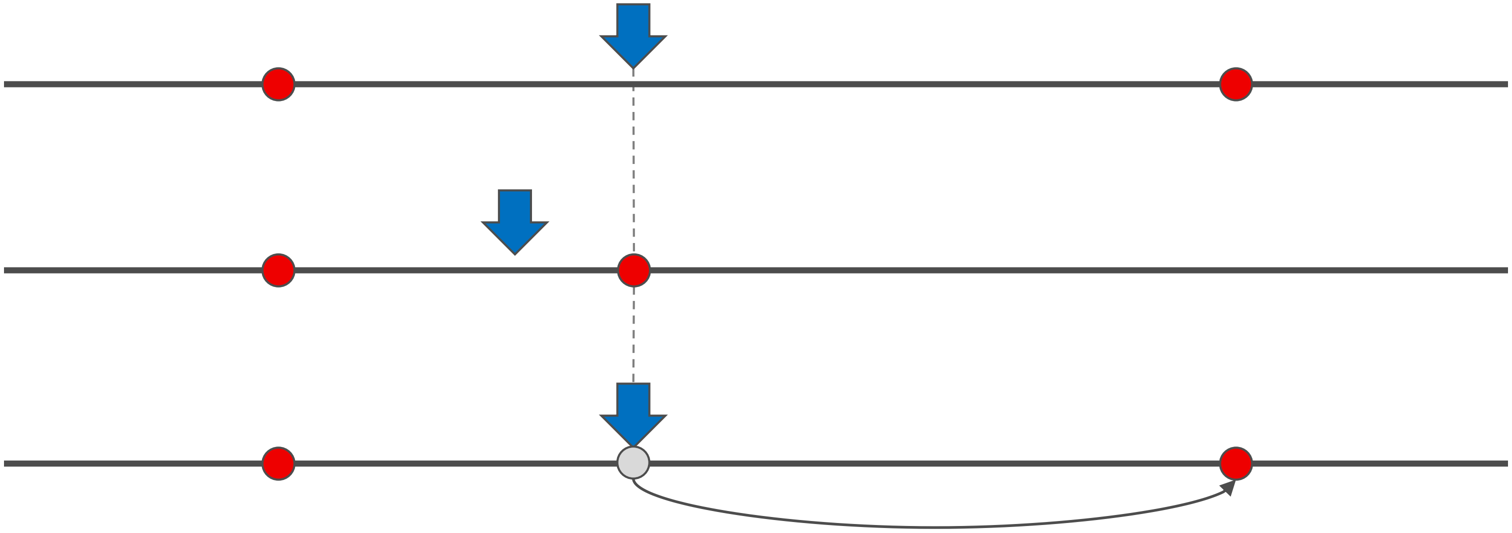
Poll 1

Approximation ratio of the median to max cost?

- In [1,2)
- In [2,3)
- In [3,4)
- In $[4, \infty)$

DETERMINISTIC RULES FOR MC

- Theorem [P and Tennenholtz 2009]: No deterministic SP rule has an approximation ratio < 2 to the max cost
- Proof:



RANDOMIZED RULES FOR MC

- We say that a randomized rule f gives an α-approximation to the max cost if for all $\boldsymbol{x} \in \mathbb{R}^n$,
$$\mathbb{E}[\mathrm{mc}(f(\boldsymbol{x}), \boldsymbol{x})] \leq \alpha \cdot \min_{y \in \mathbb{R}} \mathrm{mc}(y, \boldsymbol{x})$$
- The Left-Right-Middle (LRM) rule: Choose min x_i with prob. ¼, max x_i with prob. ¼, and their average with prob. ½

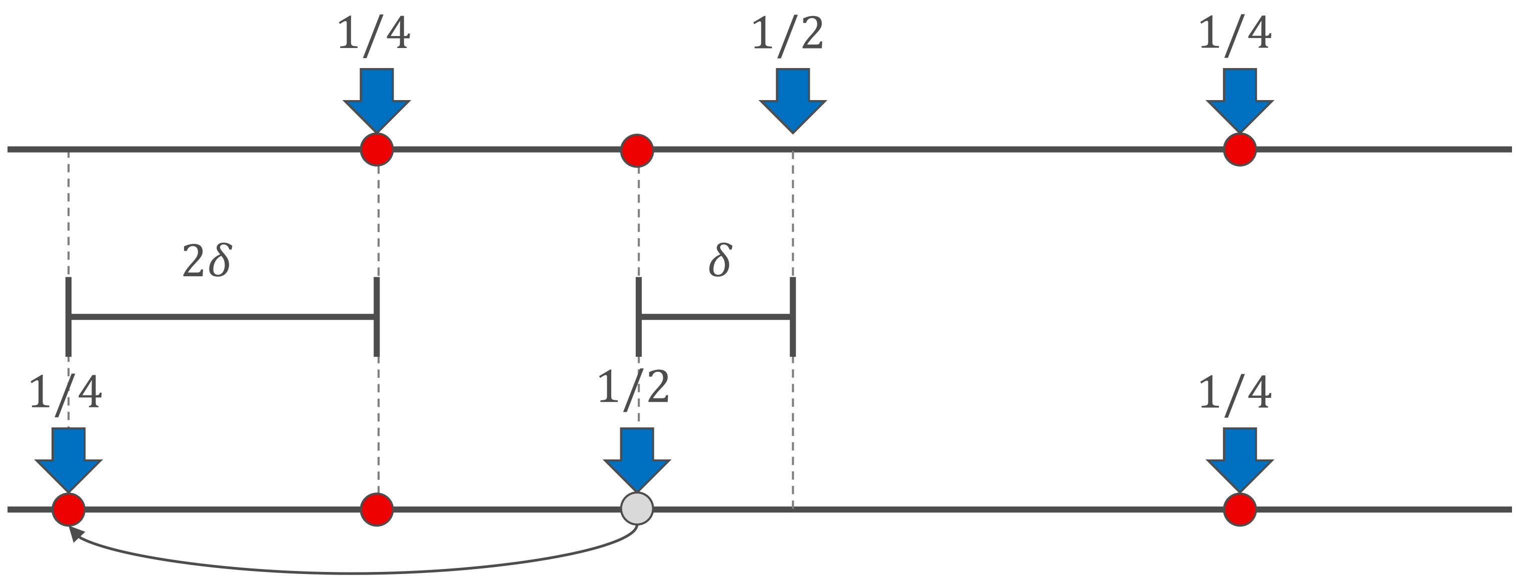
Poll 2

Approximation ratio of LRM to max cost?

- 5/4
- 6/4 = 3/2
- 7/4
- 8/4 = 2

RANDOMIZED RULES FOR MC

- Theorem [P and Tennenholtz 2009]: LRM is SP (in expectation)
- Proof:



RANDOMIZED RULES FOR MC

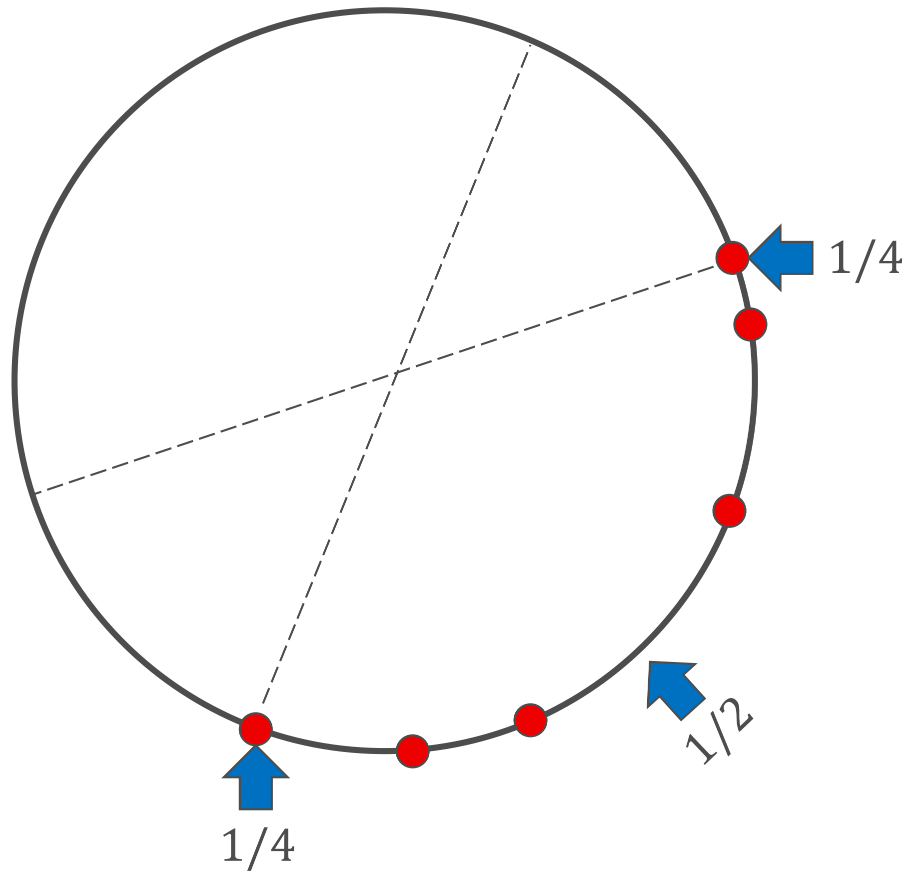
- **Theorem [P and Tennenholtz 2009]:** No randomized SP mechanism has an approximation ratio $< 3/2$
- **Proof:**
  - $x_1 = 0, x_2 = 1, f(\boldsymbol{x}) = P$
  - $\text{cost}(P, x_1) + \text{cost}(P, x_2) \geq 1$; wlog $\text{cost}(P, x_2) \geq 1/2$
  - $x_1 = 0, x_2' = 2$
  - By SP, the expected distance from $x_2 = 1$ is at least ½
  - Expected max cost at least $3/2$, because for every $y \in \mathbb{R}$, the expected cost is $|y - 1| + 1$ ∎

FROM LINES TO CIRCLES

- Continuous circle
- $d(\cdot)$ is the distance on the circle
- Assume that the circumference is 1
- “Applications”:
  - Telecommunications network with ring topology
  - Scheduling a daily task

RULES ON A CYCLE

- Semicircle like an interval on a line
- If all agents are on one semicircle, can apply LRM
- Problematic otherwise



RANDOM POINT

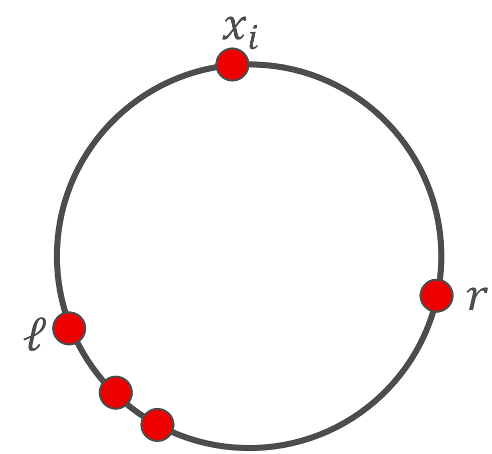
- Random Point (RP) Rule: Choose a random point on the circle
- Obviously horrible if players are close together
- Gives a 7/4 approx if the players cannot be placed on one semicircle
  - Worst case: many agents uniformly distributed over slightly more than a semicircle
  - If the rule chooses a point outside the semicircle (prob. 1/2), exp. max cost is roughly 1/2
  - If the rule chooses a point inside the semicircle (prob. 1/2), exp. max cost is roughly 3/8

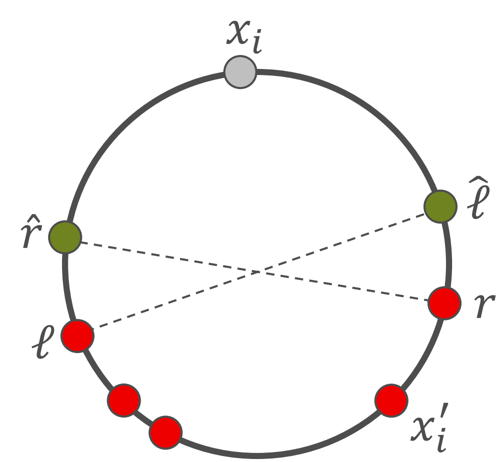
A HYBRID RULE

- Hybrid Rule 1: Use LRM if players are on one semicircle, RP if not
- Gives a 7/4 approx
- Surprisingly, Hybrid rule 1 is also SP!

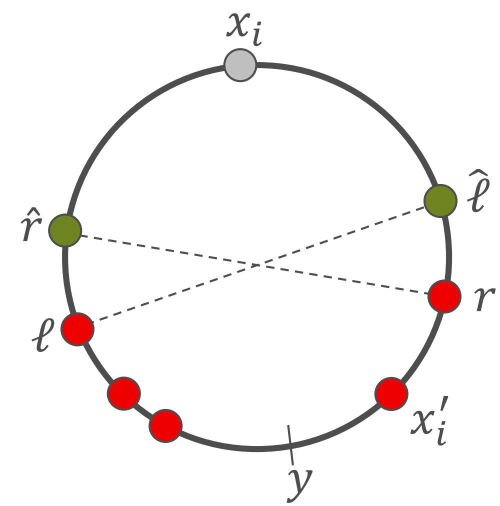
HYBRID RULE 1 IS SP

- Deviation where RP or LRM is used before and after is not beneficial
- LRM to RP: expected cost of i is at most 1/4 before, exactly 1/4 after; focus on RP to LRM
- ℓ and r are extreme locations in new profile, $\hat{\ell}$ and $\hat{r}$ their antipodal points
- Because agents were not on one semicircle in $\boldsymbol{x}$, $x_i \in (\hat{\ell}, \hat{r})$





HYBRID RULE 1 IS SP

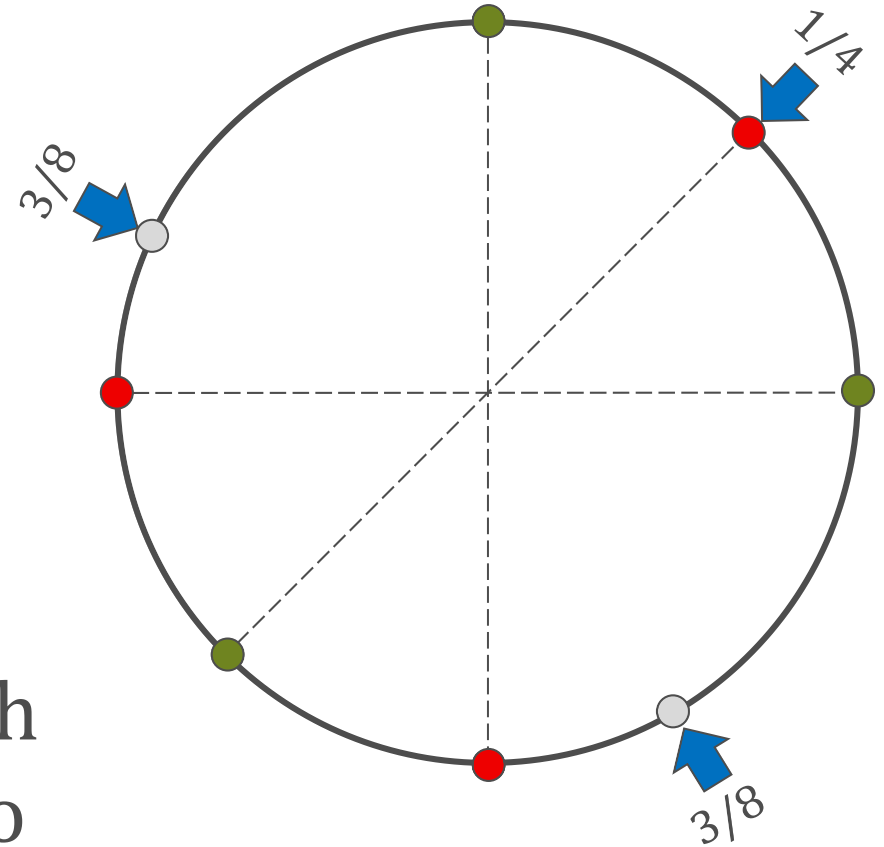


- $y =$ center of (ℓ, r)
- $d(x_i, y) \geq 1/4$, because $d(\hat{\ell}, y) \geq 1/4$, $d(\hat{r}, y) \geq 1/4$, and $x_i \in (\hat{\ell}, \hat{r})$
- Hence,

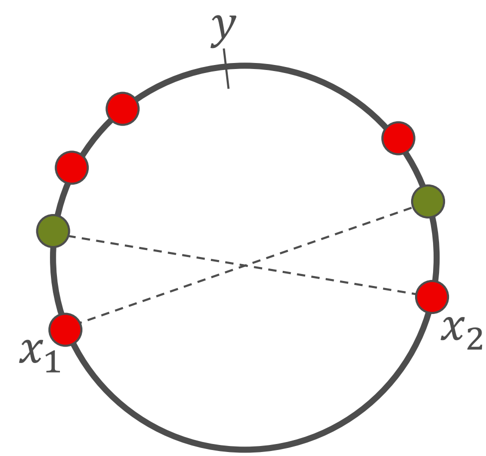
$$\begin{aligned}\mathrm{cost}(\mathrm{lrm}(\boldsymbol{x}'), x_i) &= \frac{1}{4} d(x_i, \ell) + \frac{1}{4} d(x_i, r) + \frac{1}{2} d(x_i, y) \\ &\geq \frac{1}{4}\big(d(x_i, \ell) + d(x_i, r)\big) + \frac{1}{2} \cdot \frac{1}{4} \\ &\geq \frac{1}{4} = \mathrm{cost}(\mathrm{rp}(\boldsymbol{x}), x_i) \quad \blacksquare\end{aligned}$$

RANDOM MIDPOINT

- Goal: improve the approx ratio of Hybrid 1?
- Random Midpoint (RM) Rule: choose midpoint of arc between two antipodal points with prob. proportional to length



RANDOM MIDPOINT



- **Lemma:** When the players are not on a semicircle, RM gives a 3/2 approx
- **Proof:**
  - $\alpha =$ length of the longest arc between two adjacent players, w.l.o.g. x_1 and x_2
  - $\alpha \leq 1/2$ because otherwise players are on one semicircle
  - Opt y at center of $\hat{x}_1$ and $\hat{x}_2$, so $\mathrm{OPT} = (1-\alpha)/2$
  - RM selects y with probability α, and a solution with cost at most $1/2$ with prob. $1-\alpha$
  - $\dfrac{\alpha\frac{1-\alpha}{2}+\frac{1-\alpha}{2}}{\frac{1-\alpha}{2}} = 1+\alpha \leq \frac{3}{2}$ $\blacksquare$

ANOTHER HYBRID RULE

- Hybrid Rule 2: Use LRM if players are on one semicircle, RM if not
- Theorem [Alon et al., 2010]: Hybrid Rule 2 is SP and gives a 3/2 approx to the max cost
- The proof of SP is a rather tedious case analysis… but the fact that it's SP is quite amazing!