TRUTH JUSTICE ALGOS

Social Choice II: Implicit Utilitarian Voting

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REMINDER: THE VOTING MODEL

- Set of voters $N = \{1, \ldots, n\}$
- Set of alternatives A; denote $|A| = m$
- Each voter has a ranking $\sigma_i \in \mathcal{L}$ over the alternatives; $x \succ_i y$ means that voter i prefers x to y
- A preference profile $\boldsymbol{\sigma} \in \mathcal{L}^n$ is a collection of all voters' rankings
- A voting rule is a function $f: \mathcal{L}^n \to A$

UTILITIES AND WELFARE

- The voting model assumes ordinal preferences, but it is plausible that they are derived from underlying cardinal preferences
- Assume that each voter i has a utility function $u_i: A \to [0,1]$, such that $\sum_{x \in A} u_i(x) = 1$
- Voter i reports a ranking σ_i that is consistent with his utility function, denoted $u_i \rhd \sigma_i$:
$$x \succ_i y \Rightarrow u_i(x) \geq u_i(y)$$
- As usual, the (utilitarian) social welfare of $x \in A$ is $\mathrm{sw}(x, \boldsymbol{u}) = \sum_{i \in N} u_i(x)$
- Our goal is choose an alternative that maximizes social welfare, even though we cannot observe the utilities directly

DISTORTION

- We want to quantify how much social welfare a voting rule loses due to lack of information
- The distortion of voting rule f on $\boldsymbol{\sigma}$ is

$$\text{dist}(f, \boldsymbol{\sigma}) = \max_{\boldsymbol{u} \rhd \boldsymbol{\sigma}} \frac{\max_{x \in A} \text{SW}(x, \boldsymbol{u})}{\text{SW}(f(\boldsymbol{\sigma}), \boldsymbol{u})}$$

- The distortion of voting rule f is

$$\text{dist}(f) = \max_{\boldsymbol{\sigma}} \text{dist}(f, \boldsymbol{\sigma})$$

DISTORTION

- Consider the preference profile

1	2	3
a	a	b
c	c	a
b	b	c

Poll 1

Distortion of Borda count on this profile?

- 3/2
- 2
- 5/3
- 5/2

DISTORTION

- Consider the preference profile

1	2	...	$m-1$
a_1	a_2	...	a_{m-1}
x	x	...	x
$\vdots$	$\vdots$	...	$\vdots$

Poll 2

Distortion of plurality on this profile?

- $\Theta(1)$
- $\Theta(\sqrt{m})$
- $\Theta(m)$
- $\Theta(m^2)$

DETERMINISTIC LOWER BOUND

- Theorem: Any deterministic voting rule f has distortion at least m
- Proof:
  - Partition N into two subsets with $|N_k| = n/2$, and let the profile $\boldsymbol{\sigma}$ be such that voters in N_1 rank a_1 first, and voter in N_2 rank a_2 first
  - W.l.o.g. $f(\boldsymbol{\sigma}) = a_1$
  - Let $u_i(a_2) = 1, u_i(a_j) = 0$ for $i \in N_2, u_i(a_j) = 1/m$ for all $i \in N_1$
  - It holds that

$$\text{dist}(f, \boldsymbol{\sigma}) \geq \frac{\frac{n}{2}}{\frac{n}{2m}} = m \quad \blacksquare$$

RANDOMIZED UPPER BOUND

- Under the harmonic scoring rule, each voter gives $1/k$ points to alternative ranked k-th
- Denote the score of x under $\boldsymbol{\sigma}$ as $\mathrm{sc}(x, \boldsymbol{\sigma})$
- Why is this useful? Because
$$\mathrm{sw}(x, \boldsymbol{u}) \leq \mathrm{sc}(x, \boldsymbol{\sigma})$$
for any $\mathbf{u} \rhd \boldsymbol{\sigma}$
- Theorem [Caragiannis et al. 2015]: The randomized voting rule that, with prob. ½, selects $x \in A$ with prob. proportional to $\mathrm{sc}(x, \boldsymbol{\sigma})$, and selects a uniformly random alternative with prob. ½, has distortion $O\left(\sqrt{m \log m}\right)$
- Discussion: In what sense is this result practical?

PROOF OF THEOREM

- Case 1: The welfare-maximizing x^* satisfies
$$\mathrm{sw}(x^*, \boldsymbol{u}) \geq n\sqrt{(\ln m + 1)/m}$$
- Then $\mathrm{sc}(x^*, \boldsymbol{\sigma}) \geq n\sqrt{(\ln m + 1)/m}$
- $\sum_{x \in A} \mathrm{sc}(x, \boldsymbol{\sigma}) = n \sum_{k=1}^{m} 1/k \leq n(\ln m + 1)$
- x^* is selected with prob. at least
$$\frac{1}{2} \cdot \frac{n\sqrt{\frac{\ln m + 1}{m}}}{n(\ln m + 1)} = \frac{1}{2\sqrt{m\,(\ln m + 1)}}$$
- Now,
$$\mathbb{E}[\mathrm{sw}(f(\boldsymbol{\sigma}), \boldsymbol{u})] \geq \Pr[f(\boldsymbol{\sigma}) = x^*]\,\mathrm{sw}(x^*, \boldsymbol{u})$$
$$\geq \frac{1}{2\sqrt{m\,(\ln m + 1)}}\mathrm{sw}(x^*, \boldsymbol{u})$$

PROOF OF THEOREM

- Case 2: For every $x \in A$ it holds that
$$\mathrm{sw}(x, \boldsymbol{u}) < n\sqrt{(\ln m + 1)/m}$$
- Uniformly random selection gives expected social welfare
$$\frac{1}{2}\frac{1}{m}\sum_{x \in A}\sum_{i \in N} u_i(x) = \frac{1}{2}\frac{1}{m}\sum_{i \in N}\left(\sum_{x \in A} u_i(x)\right) = \frac{n}{2m}$$
- Distortion is at most
$$\frac{\mathrm{sw}(x^*, \boldsymbol{u})}{\mathbb{E}[\mathrm{sw}(f(\boldsymbol{\sigma}), \boldsymbol{u})]} \leq \frac{n\sqrt{\frac{\ln m + 1}{m}}}{\frac{n}{2m}} = 2\sqrt{m(\ln m + 1)} \quad \blacksquare$$

RANDOMIZED LOWER BOUND

- **Theorem [Caragiannis et al. 2012]:** Any randomized voting rule f has distortion $\Omega(\sqrt{m})$
- **Proof:**
  - Partition N into subsets with $|N_k| = n/\sqrt{m}$, and let the profile be

N_1	N_2	…	$N_{\sqrt{m}}$
a_1	a_2	…	$a_{\sqrt{m}}$
⋮	⋮	⋮	⋮

  - W.l.o.g. a_1 is selected with prob. $\leq \frac{1}{\sqrt{m}}$
  - Let $u_i(a_1) = 1, u_i(a_j) = 0$ for $i \in N_1, u_i(a_j) = 1/m$ otherwise
  - $n/\sqrt{m} \leq \mathrm{sw}(a_1, \boldsymbol{u}) \leq 2n/\sqrt{m}$, whereas $\mathrm{sw}(a_j, \boldsymbol{u}) \leq n/m$ for $j \neq 1$
  - Distortion is at least

$$\frac{\frac{n}{\sqrt{m}}}{\frac{1}{\sqrt{m}} \cdot \frac{2n}{\sqrt{m}} + \left(1 - \frac{1}{\sqrt{m}}\right) \cdot \frac{n}{m}} \geq \frac{\sqrt{m}}{3} \quad \blacksquare$$

PARTICIPATORY BUDGETING

Porto Alegre
Brazil
Since 1989

Paris
France
€100M (2016)

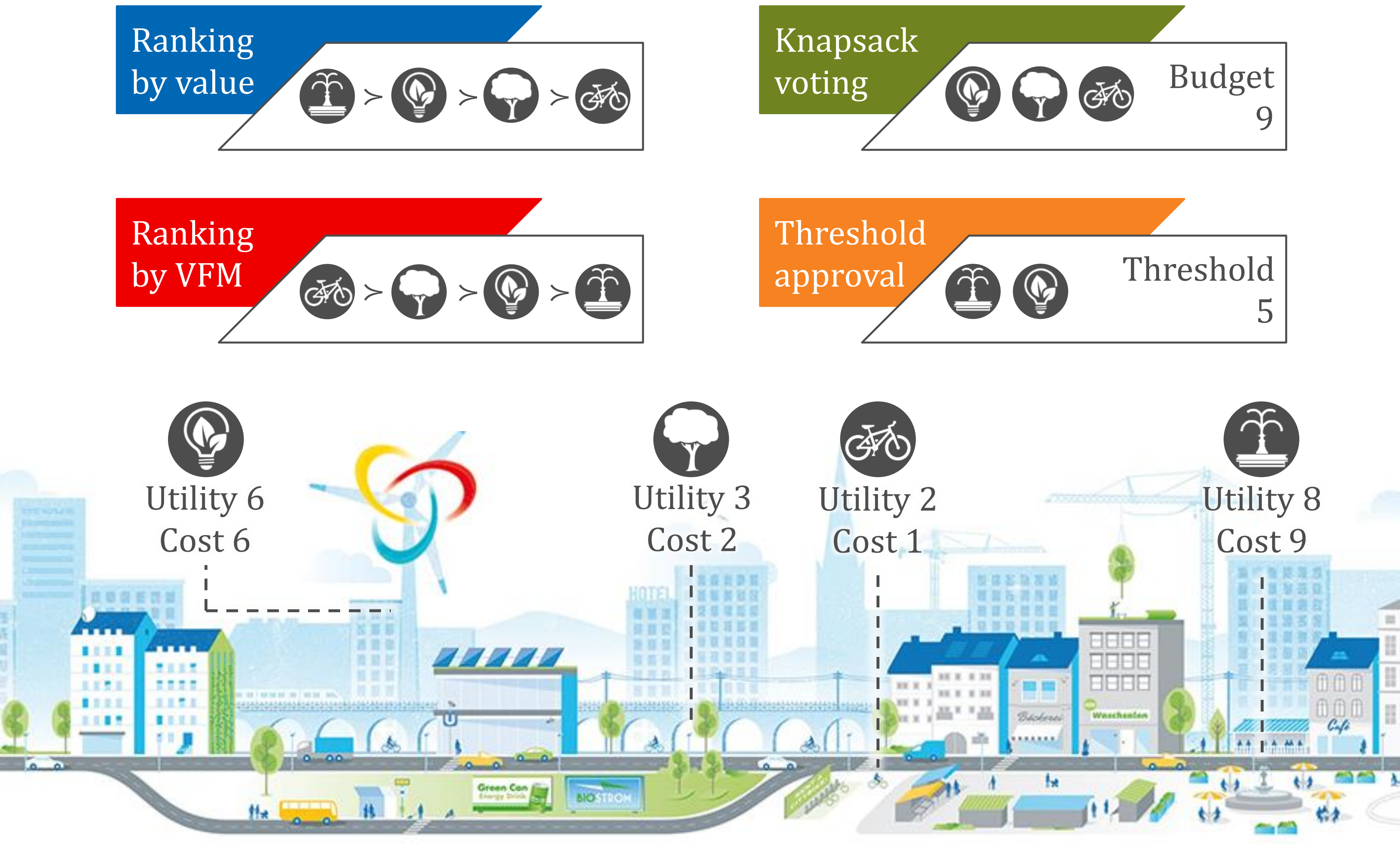
Madrid
Spain
€24M (2016)

New York
USA
$40M (2017)

THE MODEL

- The total budget is B
- Each alternative x has a cost c_x
- For $X \subseteq A$, the cost $c(X)$ is additive
- Utilities are also additive, that is, $u_i(X) = \sum_{x \in X} u_i(x)$
- The goal is to find $X \subseteq A$ that maximizes the social welfare $\mathrm{sw}(X, \boldsymbol{u}) = \sum_{i \in N} u_i(X)$ subject to the budget constraint $c(X) \leq B$

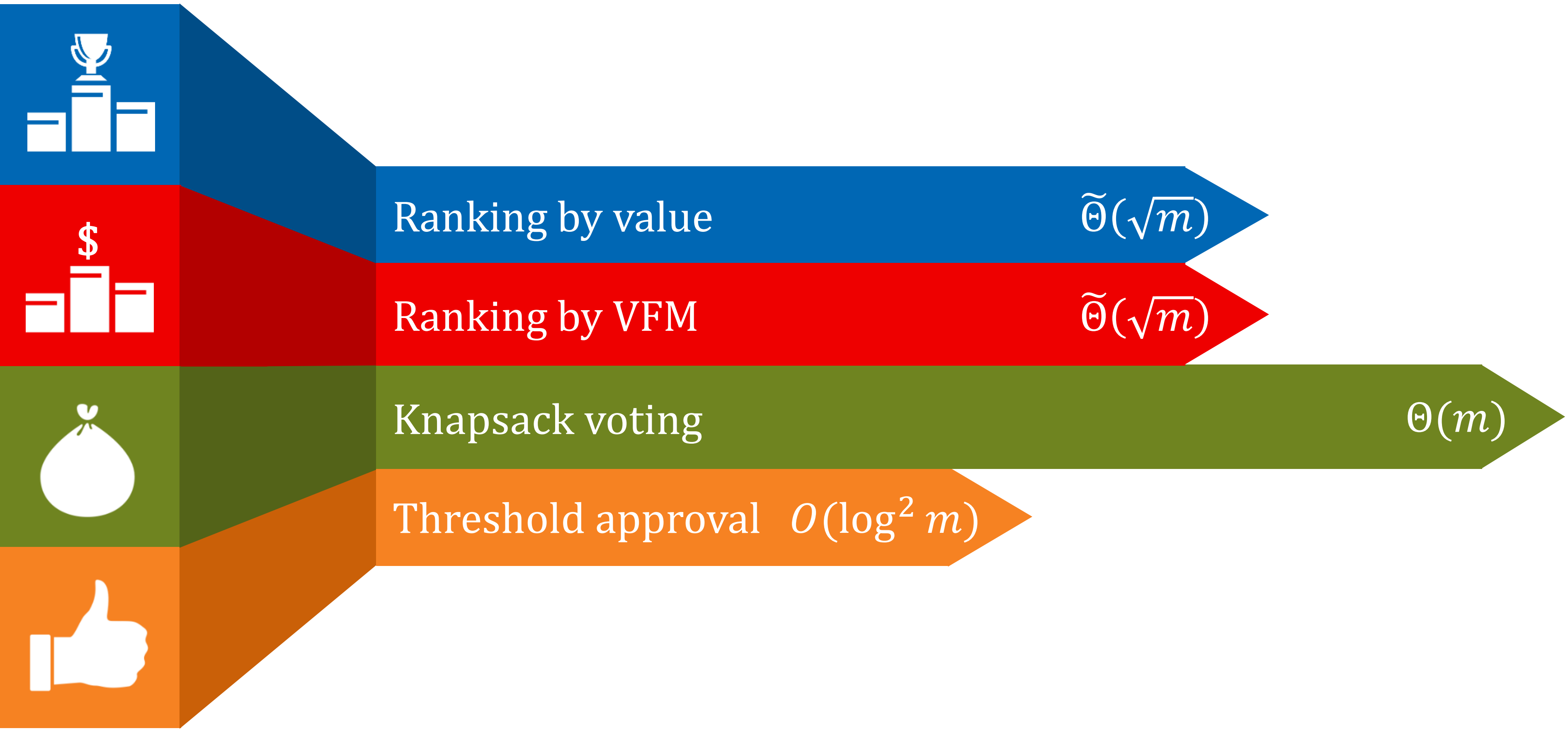
INPUT FORMATS



DISTORTION REDUX

- Distortion allows us to objectively compare input formats, by associating an input format with the distortion of the best voting rule
- Theorem [Benade et al. 2017]: Any randomized voting rule has distortion at least $\Omega(m)$ under knapsack votes
- Proof:
    - Let $B = 1, c(a_j) = 1$ for all $a_j \in A$
    - Define $\boldsymbol{\sigma}$: For each $a_j \in A$ we have n/m voters N_j who choose x
    - W.l.o.g. a_1 is selected with prob. $\leq 1/m$, then let $u_i(a_1) = 1$ for all $i \in N_1$, and $u_i(a_j) = u_i(a_1) = 1/2$ for all $i \in N_j, j \neq 1$ ■

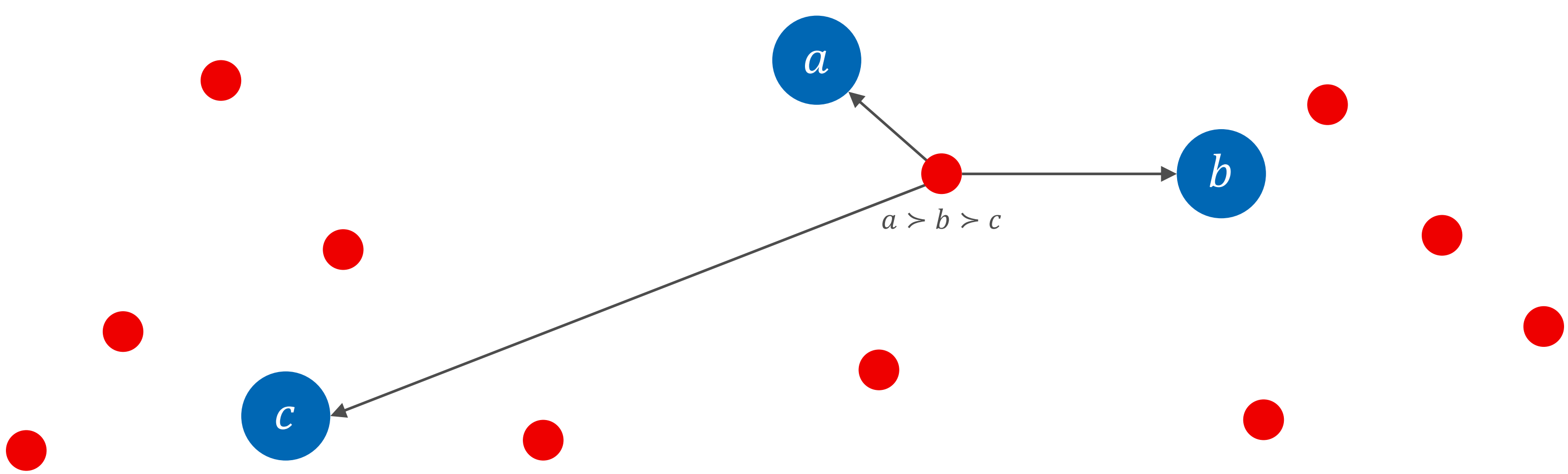
RANDOMIZED BOUNDS



[Benade et al., 2017]

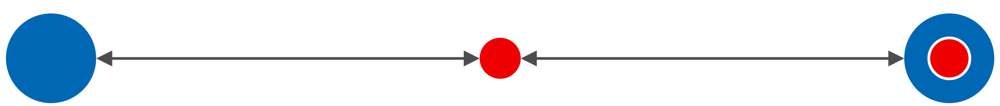
METRIC PREFERENCES

- Assume a metric space with metric d on space of voters and alternatives
- Preferences are defined by
$$d(i,x) < d(i,y) \Rightarrow x \succ_i y$$
- Now we want to minimize the social cost, defined as $\mathrm{sc}(x,d) = \sum_{i \in N} d(i,x)$



LOWER BOUND

- Theorem [Anshelevich et al. 2015]: The distortion of any deterministic rule under metric preferences is at least 3
- Proof:



- Theorem [Anshelevich et al. 2015]: The distortion of Copeland under metric preferences is at most 5