TRUTH JUSTICE ALGOS

Mechanism Design: Recent Advances

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SO FAR

- Revelation Principle
- Single parameter environments
  - Second price auctions
  - Myerson's lemma
  - Myerson's optimal auction
  - Cremer-McLean auction for correlated buyers
  - Prophet inequalities
  - Bulow-Klemperer
- Multiparameter environments
  - The VCG mechanism
  - Challenges
  - Revenue optimal auctions are strange

TODAY

- Computing the optimal auction
  - Reduced forms
- Simple vs Optimal mechanisms
  - $SRev$ and $BRev$ are not good approximations
  - $\max\{SRev, BRev\}$ is
  - Langrangian duality
- Dynamic mechanisms

CAN WE COMPUTE STUFF FOR MANY BIDDERS?

- Assume that buyers are additive over items.
- DSIC: Too many constraints to even write down!
- Standard approach: BIC (Bayesian Incentive Compatible)
  - "If everyone is telling the truth, bidding my true values is the optimal strategy"

$$\sum_{v_{-i} \sim V_{-i}} \Pr[V_{-i} = v_{-i}]\left(\sum_j v_{ij} x_{ij}(\vec{v}) - p_i(\vec{v})\right)$$

$$\geq \sum_{v_{-i} \sim V_{-i}} \Pr[V_{-i} = v_{-i}]\left(\sum_j v_{ij} x_{ij}(\vec{v}') - p_i(\vec{v}')\right)$$

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- n bidders, m items, $|V_i|$ =support of V_i

Poll

How many variables?

1. $\Theta(nm \prod_i |V_i|)$
2. $\Theta(n^m \sum_i |V_i|)$
3. $\Theta(\sum_i |V_i|)$
4. Beats me

CAN WE COMPUTE STUFF FOR MANY BIDDERS?

- Reduced form

$\pi_{ij}(\overrightarrow{v_i}) = \Pr[\text{item } j \text{ goes to } i \text{ if she reports } \overrightarrow{v_i}\,]$

  - “Interim allocation rule”

- BIC:

$$\sum_j v_{ij}\pi_{ij}(\overrightarrow{v_i}) - p_i(\overrightarrow{v_i}) \geq \sum_j v_{ij}\pi_{ij}\left(\overrightarrow{v_i{}'}\right) - p_i(\overrightarrow{v_i{}'})$$

- Down to $\Theta(nm \cdot max_i|V_i|)$ variables and constraints!
- New problem: How do we know that there is an auction that corresponds to a given reduced form?

REDUCED FORMS

- One item, two bidders: $V_1 = U\{A, B, C\}$, $V_2 = U\{D, E, F\}$
- Question: Is the following r.f. feasible?

$$\pi_{11}(A) = 1 \qquad \pi_{21}(D) = 2/3$$
$$\pi_{11}(B) = 1/2 \qquad \pi_{21}(E) = 5/9$$
$$\pi_{11}(C) = 0 \qquad \pi_{21}(F) = 0$$

- $(A, D/E/F) \rightarrow 1$ wins ($\pi_{11}(A) = 1$)
- $(B/C, D) \rightarrow 2$ wins ($\pi_{21}(D) = 2/3$)
- $(B, F) \rightarrow 1$ wins (1/3 out of 1/2, 1/6 to go)
- $(C, E) \rightarrow 2$ wins (1/3 out of 5/9, 2/9 to go)
- $(B, E) \rightarrow$???
  - B needs to win with probability 1/2
  - E needs to win with probability 2/3

IMPOSSIBLE

REDUCED FORMS

- Can we check if a reduced form is feasible quickly?
- Border's theorem: The following a necessary and sufficient condition of a reduced form to be feasible. For every item j and every $S_1 \subseteq V_1, \dots, S_n \subseteq V_n$

$$\sum_{i \in [n]} \sum_{\vec{v}_i \in S_i} \Pr[\vec{v}_i]\, \pi_i(\vec{v}_i) \leq 1 - \prod_{i \in [n]} (1 - \sum_{\vec{v}_i \in S_i} \Pr[\vec{v}_i])$$

- LHS = Probability that winner has value in S_i
- RHS = Probability that there is someone with value in S_i

REDUCED FORMS

- For every item *j* and every $S_1 \subseteq V_1, \ldots, S_n \subseteq V_n$

$$\sum_{i \in [n]} \sum_{\vec{v}_i \in S_i} \Pr[\vec{v}_i]\, \pi_i(\vec{v}_i) \leq 1 - \prod_{i \in [n]} \left(1 - \sum_{\vec{v}_i \in S_i} \Pr[\vec{v}_i]\right)$$

- That's $2^{\sum_i |V_i|}$ conditions!
- [CDW'12]: We can check feasibility in time almost linear in $\sum_i |V_i|$
  - Key result in solving the succinct LP.

For the remaining we focus on the case of a single additive buyer with m independent items

CHARACTERIZATIONS OF THE OPTIMAL MECHANISM

- When is the revenue maximizing auction "nice", even for a single buyer?
- For example, when is it optimal to post a price for the grand-bundle?
  - Grand-bundle = all the items as a single bundle
- There are necessary and sufficient conditions! [DDT 15]
- Unfortunately, these conditions are not very intuitive
  - Measure theory conditions
- Very interesting outcomes though:
  - For every number of items m, there exists a c, such that the optimal mechanism for m i.i.d. $U[c, c+1]$ items is a grand-bundling mechanism
  - On the other hand, for every c, there exists a number m_0, such that for all $m > m_0$, the grand-bundle mechanism is **not** optimal for m i.i.d. $U[c, c+1]$ items!

SIMPLE AND APPROXIMATELY OPTIMAL MECHANISMS

- Is selling only the grand bundle a good (constant) approximation to the optimal mechanism?
- No!
  - Not even a good approximation to $SRev$

$BRev$ VS OPT

Example:

- $v_i \in \{0, M^i\}$, where M is a large number
- $\Pr\left[v_i = M^i\right] = 1/M^i$
- $Rev(D_i) = 1$
  - So, $SRev = m$
- $BRev \leq \max_k M^k \cdot \Pr[\sum_j v_j \geq M^k]$
  - $\Pr[\sum_j v_j \geq M^k] \leq \sum_{j \geq k} \Pr\left[v_j = M^j\right] = M^{-j}$
  - $\sum_{j \geq k} M^{-j} = M^{1-k}/(M-1)$
- $BRev \leq 1 + 1/(M-1)$

SIMPLE AND APPROXIMATELY OPTIMAL MECHANISMS

- Is selling each item separately a good (constant) approximation to the optimal mechanism?
- No!
  - Example a bit too complicated…
  - m i.i.d. items from a “equal revenue” distribution: $F(x) = 1 - 1/x$

SIMPLE AND APPROXIMATELY OPTIMAL MECHANISMS

- What about the best of $SRev$ and $BRev$?
- Theorem [BILW 14]:

$$\max\{SRev, BRev\} \geq \frac{1}{6} Rev$$

- Some definitions
  - m = number of items
  - V_j random variable for the value of item j
  - $f_j(v_j) = \Pr[V_j = v_j]$
  - $R_j = \{\vec{v} : v_j \geq v_k, \forall k \in [m]\}$
    - Set of profiles where j is the favorite item

PROOF SKETCH

- Two parts:

1. $Rev \leq Benchmark$
2. $Benchmark \leq 6\max\{SRev, BRev\}$

- Today: Part 1

A DETOUR: LAGRANGIAN DUALITY

- Optimization

$$\max x_1 + 3x_2 + 5x_3$$

Subject to

$$x_2 + x_3 \leq 10$$

$$x_1 \leq 2$$

$$\ldots$$

- Lagrangian function

$$\mathcal{L}(x, \lambda) = x_1 + 3x_2 + 5x_3 + \lambda(10 - x_2 - x_3)$$

A DETOUR: LAGRANGIAN DUALITY

- Lagrangian function
  $\mathcal{L}(x, \lambda) = x_1 + 3x_2 + 5x_3 + \lambda(10 - x_2 - x_3)$
- Let OPT be the optimal solution to the optimization problem
- Game:
  - We pick $\lambda \geq 0$
  - Adversary picks $x_1, \ldots$ that satisfy all the constraints ***except*** the one we "Lagrangified" in order to maximize $\mathcal{L}(\vec{x}, \lambda)$
- Theorem: $\forall \lambda \geq 0, OPT \leq \max_{\vec{x}} \mathcal{L}(\vec{x}, \lambda)$

A DETOUR: LAGRANGIAN DUALITY

- Lagrangian function
  $\mathcal{L}(x, \lambda) = x_1 + 3x_2 + 5x_3 + \lambda(10 - x_2 - x_3)$
- Intuition:
  - If $\lambda = 0$, then it's as if we dropped that constraint
  - If $\lambda = \infty$, if we violate the Lagrangified constraint we pay an infinite penalty. But, if we strictly satisfy it we get a bonus

A DETOUR: LAGRANGIAN DUALITY

- Why would this be useful?
- Sometimes you know how to solve a problem if you “remove” a constraint
  - Canonical example: Find the shortest path between s and t, that also uses at most k edges
    - Lagrangify the “at most k edges” constraint.

BACK TO REVENUE

- For now, single buyer
- Objective:

$$\max \sum_{v \in V} p(v) \cdot \Pr[value = v]$$

- Constraints:
  - IC: $\forall v, v' \in V: vx(v) - p(v) \geq vx(v') - p(v')$
  - IR: $\forall v \in V: vx(v) - p(v) \geq 0$
  - Feasibility: $\forall v \in V: 1 \geq x(v) \geq 0$

REVENUE

$$\max \sum_{v \in V} p(v) \cdot f(v)$$
$$\forall v \in V, v' \in V \cup \{\bot\}: vx(v) - p(v) \geq vx(v') - p(v')$$
$$\forall v \in V: 1 \geq x(v) \geq 0$$

- Lagrangify the IC+IR constraint!

$$\mathcal{L} = \sum_{v \in V} f(v)p(v) +$$
$$\sum_{v \in V} \sum_{v' \in V \cup \{\bot\}} \lambda(v, v') \cdot (vx(v) - p(v) - vx(v') + p(v'))$$

REVENUE

- Re-arrange:

$$\mathcal{L} = \sum_{v \in V} x(v) \left(\sum_{v' \in V \cup \{\perp\}} v\lambda(v, v') - \sum_{v' \in V} v'\lambda(v', v) \right)$$

$$+ \sum_{v \in V} p(v) \left(f(v) + \sum_{v' \in V} \lambda(v', v) - \sum_{v' \in V \cup \{\perp\}} \lambda(v, v') \right)$$

- Game:
  - We pick $\lambda(v, v') \geq 0$ for all v, v'
  - Adversary maximizes $\mathcal{L}$ subject to $x(v) \in [0,1]$
- Goal: make $\mathcal{L}^*$ as small as possible

REVENUE

$$\mathcal{L} = \sum_{v \in V} x(v) \left(\sum_{v' \in V \cup \{\perp\}} v\lambda(v, v') - \sum_{v' \in V} v' \lambda(v', v) \right)$$

$$+ \sum_{v \in V} p(v) \left(f(v) + \sum_{v' \in V} \lambda(v', v) - \sum_{v' \in V \cup \{\perp\}} \lambda(v, v') \right)$$

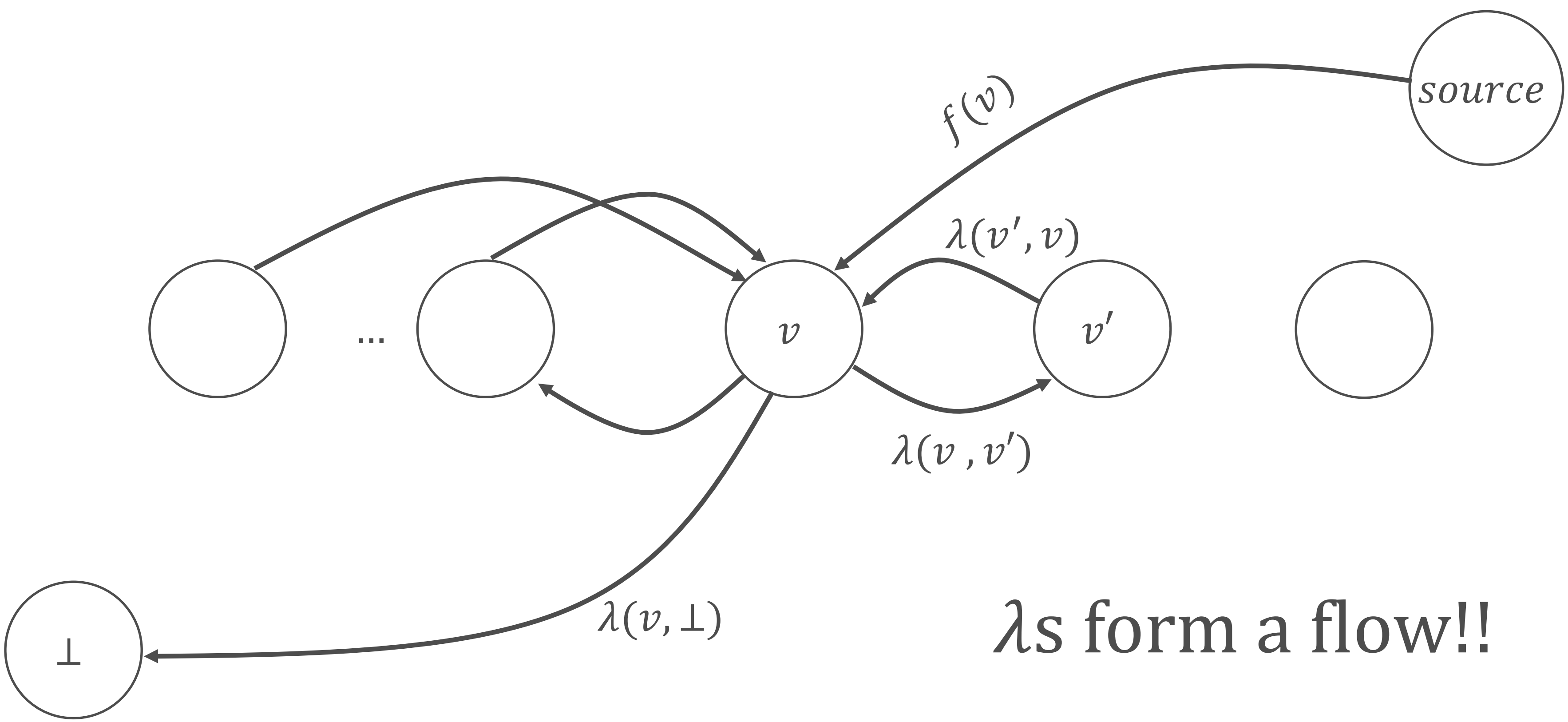
- Observation: no constraints on $p(v)$
- Therefore:

$$f(v) + \sum_{v' \in V} \lambda(v', v) - \sum_{v' \in V \cup \{\perp\}} \lambda(v, v') = 0$$

- Otherwise, $\mathcal{L}^* = \infty$!

REVENUE

$$f(v) + \sum_{v' \in V} \lambda(v', v) - \sum_{v' \in V \cup \{\perp\}} \lambda(v, v') = 0$$



λs form a flow!!

REVENUE

- Simplify:

$$\mathcal{L} = \sum_{v \in V} x(v)\left(vf(v) + \sum_{v' \in V} v\lambda(v', v) - \sum_{v' \in V} v'\lambda(v', v)\right)$$

$$= \sum_{v \in V} f(v)x(v)\left(v - \frac{1}{f(v)} \sum_{v' \in V} \lambda(v', v)(v' - v)\right)$$

- Game:
  - We pick a **flow** λ
  - Adversary tries to maximize $\mathcal{L}(\lambda)$
- Adversary will pointwise maximize

$$\Phi(v) = v - \frac{1}{f(v)} \sum_{v' \in V} \lambda(v', v)(v' - v)$$

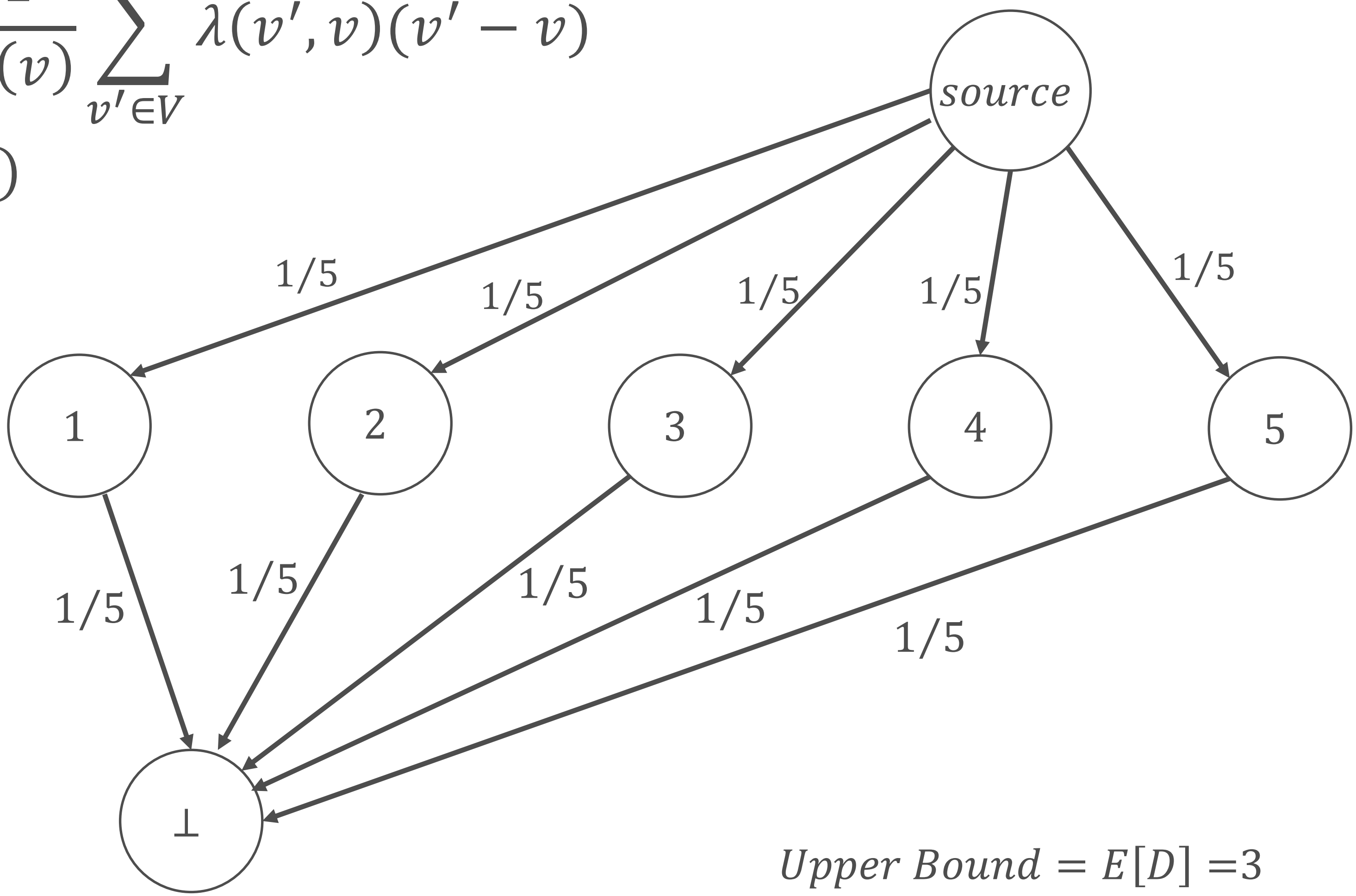
EXAMPLE

- $D = U\{1,2,3,4,5\}$

$$\Phi(v) = v - \frac{1}{f(v)} \sum_{v' \in V} \lambda(v', v)(v' - v)$$

$$\lambda(v, \perp) = f(v)$$

$$\Phi(v) = v$$

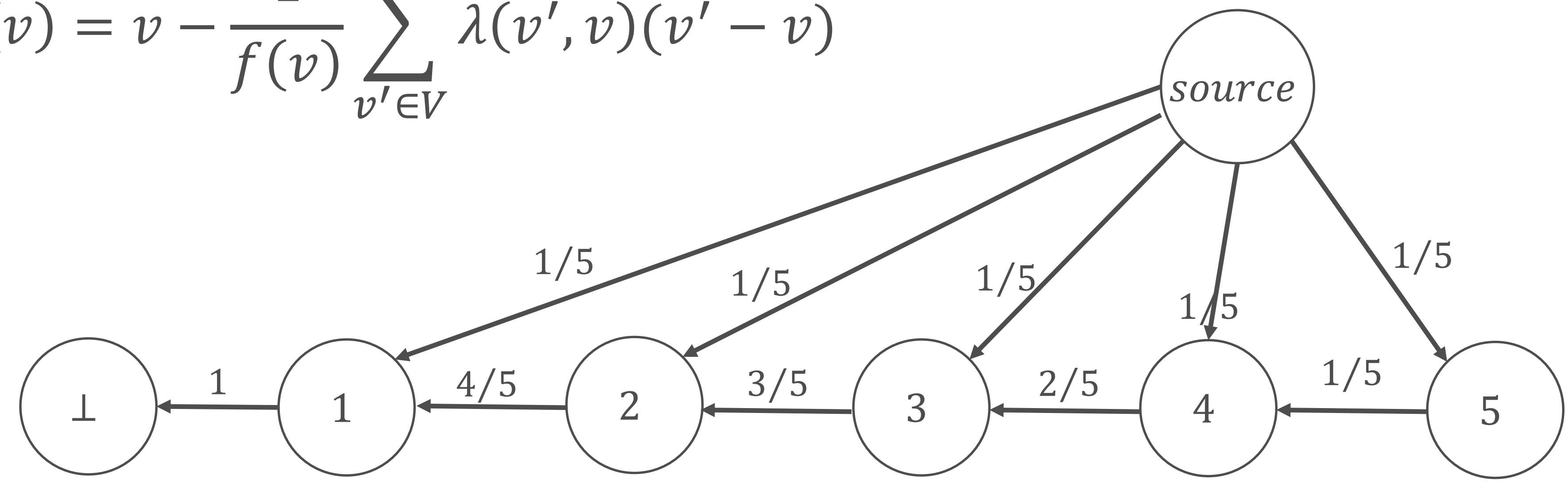


$Upper\ Bound = E[D] = 3$

EXAMPLE

- $D = U\{1,2,3,4,5\}$

$$\Phi(v) = v - \frac{1}{f(v)} \sum_{v' \in V} \lambda(v', v)(v' - v)$$



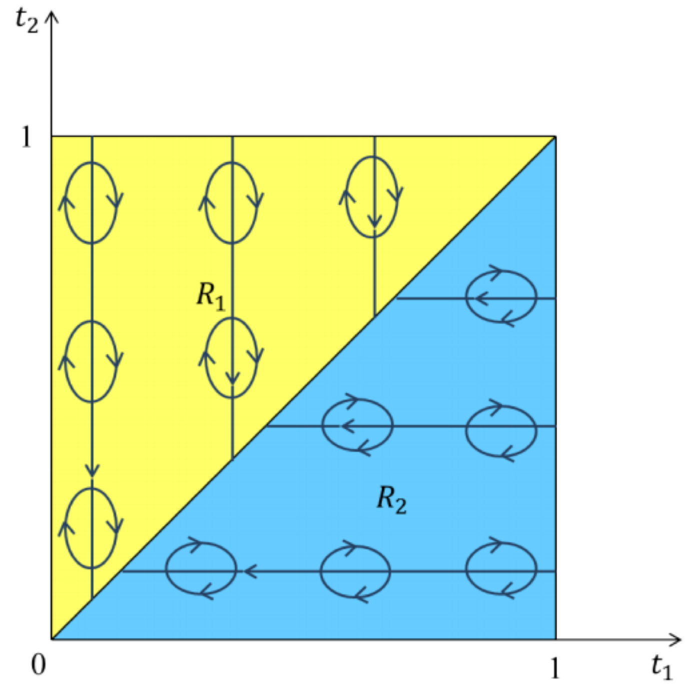
- $\Phi(5) = 5$
- $\Phi(4) = 4 - \frac{1}{1/5} \cdot \frac{1}{5} \cdot (5 - 4) = 3$
- $\Phi(3) = 3 - \frac{1}{1/5} \cdot \frac{2}{5} \cdot (4 - 3) = 1$
- $\Phi(2) = 2 - \frac{1}{1/5} \cdot \frac{3}{5} \cdot (3 - 2) = -1$
- $\Phi(1) = 1 - \frac{1}{1/5} \cdot \frac{4}{5} \cdot (2 - 1) = -3$

$$Upper\ Bound = \frac{5 + 3 + 1}{5} = \frac{9}{5}$$

What's OPT?

PROOF SKETCH

- Same idea for many items
- Have to find a good “flow”



PROOF SKETCH

- Lemma 1: Rev is at most

$$\sum_{\vec{v}}\sum_{j} f(\vec{v}) \cdot x_j(\vec{v}) \cdot \phi_j(v_j) \cdot \mathbb{I}\{\vec{v} \in R_j\} \ (SINGLE)$$

$$+\sum_{\vec{v}}\sum_{j} f(\vec{v}) \cdot x_j(\vec{v}) \cdot v_j \cdot \mathbb{I}\{\vec{v} \notin R_j\} \ (NONFAV)$$

- Intuition:
  - SINGLE = Favorite item contributes its virtual value
  - NONFAV = Every other item contributes its value
- Theorem [BILW 14]: $\max\{SRev, BRev\} \geq \frac{1}{6} Rev$
  - Similar results exist for many buyers, even beyond additive valuation functions

DYNAMIC MECHANISMS

- Slight twist to the model
- Two items: one today, one tomorrow

Game:

- D_1, D_2 are public knowledge
- Buyer learns $v_1 \sim D_1$, submits b_1
- Item 1 and payments according to $x_1(b_1), p_1(b_1)$
- Buyer learns $v_2 \sim D_2$, submits b_2
- Item 2 and payments according to $x_2(b_1, b_2), p_2(b_1, b_2)$

DYNAMIC MECHANISMS

- When submitting b_1 buyer has to take into account how this will affect the (expected) utility she'll get from item 2
- D_1 and D_2 could be correlated
- For now assume independence
- Independent?
  - Shouldn't Myerson + Myerson be optimal?
  - Even if not optimal, it's definitely a good approximation!

DYNAMIC MECHANISMS

- $v_1 = 2^i$ with probability $2^{-i}, i = 1 \dots n$
- $v_2 = 2^i$ with probability $2^{-i}, i = 1 \dots 2^n$
  - With the remaining probability they're equal to zero

Poll

What's (roughly) $Rev(D_2)$ and $E[D_2]$?

1. n and 2^n
2. 2 and 2^n
3. 2 and n
4. 2^n and 2^n

DYNAMIC MECHANISMS

- Myerson + Myerson = constant
- Consider the following auction
  - $x_1(b_1) = 1, p_1(b_1) = b_1$
  - $x_2(b_1, b_2) = b_1/E[D_2], p_2(b_1, b_2) = 0$
  - So first day you pay your bid b_1
  - Second you get it for free w.p. $b_1/E[D_2]$
- $E[utility\ of\ reporting\ b_1]$?
  - $ut.from\ day\ 1 = v_1 - b_1$
  - $E[ut.from\ day\ 2] = \sum_{v_2} \Pr[v_2]\, v_2 \cdot \frac{b_1}{E[D_2]} = b_1$
- So, $E[ut.of\ b_1] = v_1$!
- $Rev = E[v_1] = n$

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