Due on November 5 at 11:59PM

1. **Revenue Equivalence** (25 points: 5/15/5)

   Consider a two-bidder, one-item auction. The value v_i of each bidder i is drawn independently and identically from $U[0,1]$.

   (a) What is the expected revenue of the seller, when she runs a second price auction?

   (b) Strategies $(\beta_1, \ldots, \beta_n)$ form a *Bayes-Nash equilibrium*, where $\beta_i : [0,1] \rightarrow [0,1]$ is a function that maps true values to bids, if player i maximizes her expected utility by bidding $\beta_i(v_i)$ when the remaining players play according to strategies β_{-i}, where the expectation is taken over the random values of the other players.

   Suppose the seller runs a first price auction. Let $\beta_1 = \beta_2 = \beta$ be an equilibrium strategy (the game is symmetric, so there is a symmetric equilibrium). Assume that β is differentiable, continuous and strictly increasing. So, bidder 1 with true value v_1 will bid $\beta(v_1)$ knowing that bidder 2 is bidding $\beta(V_2)$, where V_2 is the random variable indicating the true value of bidder 2. What is the function β?

   Hint: First, figure out what the expected utility of bidder 1 is when she bids b but her true value is v_1, and bidder 2 bids according to β. Then, re-write the above expression, noticing that for every $b \in [\beta(0), \beta(1)]$ there exists some w such that $\beta(w) = b$.

   (c) What is the expected revenue of the seller, when she runs a first price auction and the bidders play according to the Bayes-Nash equilibrium β?

2. **From Auctions to Order Statistics** (40 points: 10/5/15/10)

   A distribution has monotone hazard rate (MHR) if the function $h(v) = \frac{f(v)}{1-F(v)}$ is monotone non-decreasing. Let $X_{j:n}$ be the j-th order statistic of n samples: the expected j-th maximum of n samples from a random variable X. It is known that order statistics of monotone hazard rate distributions have monotone hazard rate themselves, i.e. if X is MHR, then $X_{j:n}$ is MHR, for all j and n. Finally, let $\mathbf{X}_n = \prod_{i=1}^{n} X$ denote the product distribution of n i.i.d. bidders from X.

   For the following statements, assume that X is an MHR distribution that takes only non-negative values.

   (a) Prove that $\text{Rev}(\mathbf{X}_n) \geq \text{Rev}(X_{1:n})$, i.e. the optimal revenue we can extract from n i.i.d. bidders from X is at least the optimal revenue from a single bidder from $X_{1:n}$.

   (b) Prove that the random variable $Y = -\log(1 - F(X))$ follows the exponential distribution with parameter 1. That is, show that the CDF of Y is precisely $F(y) = 1 - e^{-y}$.

   (c) Prove that $\Pr[X \geq \mathbb{E}[X]] \geq \frac{1}{e}$.

   Hint: The CDF of X can be written as $1 - e^{-A(x)}$, where A is a convex function. Also, use Jensen's inequality.

(d) Conclude that $\mathbb{E}[X_{2:n+1}] \geq \frac{1}{e}\mathbb{E}[X_{1:n}]$.
Hint: Use the Bulow-Klemperer Theorem (!!).

3. **Single-Minded Bidders** (35 points: 10/25)

Consider a multi-bidder, multi-item auction in which each bidder i values a subset S_i (as well as any of its supersets) at v_i, and any subset S_j that does not contain S_i at 0. Even in this restricted setting, the social welfare (that is, sum of values of bidders for their allocated bundles) is NP-hard to approximate to a factor of $o(\sqrt{m})$. For this reason, we cannot run VCG in polynomial time.

Now, consider the following greedy mechanism; you will prove that this algorithm is truthful and a $\sqrt{m}$-approximation of the welfare-maximizing allocation. On a high level, this algorithm sorts bids by some criterion and then allocates greedily by adding bidders to the set of winners as long as their valued subset does not intersect with any other subset that has been allocated so far.

GREEDY MECHANISM

- Initialization:
  - Reorder the bids such that $\frac{v_1}{\sqrt{|S_1|}} \geq \frac{v_2}{\sqrt{|S_2|}} \geq \cdots \geq \frac{v_n}{\sqrt{|S_n|}}$
  - $W \leftarrow \emptyset$
- For $i = 1, \ldots, n$:
  - If $S_i \cap \left(\bigcup_{j \in W} S_j\right) = \emptyset$, then $W \leftarrow W \cup \{i\}$

This outputs an allocation and payments. The set of winners is W, i.e., each $i \in W$ gets S_i. The payment for each $i \in W$ is $p_i = v_j \cdot \sqrt{|S_i|/|S_j|}$, where j is the smallest index $j \neq i$ such that $S_i \cap S_j \neq \emptyset$ and for all $k < j$, $k \neq i$, $S_j \cap S_k = \emptyset$. If no such j exists, $p_i = 0$.

(a) Prove that the greedy mechanism is truthful.

(b) Prove that the greedy mechanism provides a $\sqrt{m}$-approximation to the optimal social welfare.
Hint: The Cauchy-Schwarz inequality may be useful here: given real-valued $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$, $\sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}$.