15896/15483 Truth, Justice, and Algorithms

Fall 2018 Procaccia and Psomas HW 2

Due Thursday, October 18 at 11:59PM

1. **PoA of Awesome Games** (30 points: 5/15/10)

   Let an *awesome game* consist of a set N of n players and a set M of m activities. Each player i chooses a subset of activities $S_i \subseteq M$ from a collection of allowable subsets $\Pi_i \subseteq 2^M$, and the cost of activity j is n_j, where n_j is the number of people participating in activity j. The cost of a player i is the sum of her costs over activities she participates in; the social cost is the sum of player costs.

   (a) Prove that any awesome game has a pure-strategy Nash equilibrium.
   Hint: This problem has potential.

   (b) Prove that the price of anarchy (with respect to social cost) in awesome games is at most $5/2$.
   Hint: You may use without proof that for every pair of nonnegative integers x and y, $y(x+1) \leq \frac{1}{3}x^2 + \frac{5}{3}y^2$.

   (c) Prove that that the upper bound of part (b) is tight for any $n \geq 3$, by constructing an appropriate family of awesome games.

2. **PoA of Special Games** (30 points: 20/10)

   Consider the following class of *special games*, which are related to awesome games. Given a set N of n players, each player i has an associated weight w_i. There is also a set M of m activities, and each player would like to participate in exactly one activity. Each player chooses an activity from the set of all activities, that is, the strategy space of each agent is M. Furthermore, the cost of each player i is the sum of weights of players who choose the same activity as she did. The objective in special games is to minimize the maximum cost of any player (equivalently, minimize the maximum total weight in any activity). It is known that special games also always have pure Nash equilibria.

   (a) Prove that the price of anarchy of special games (with respect to maximum cost) is at most $2 - 2/(m+1)$.

   (b) Prove that the upper bound of part (a) is tight, by constructing an appropriate family of special games for each $m \in \mathbb{N}$.

3. **Lemke-Howson Revisited** (15 points)

   Recall that the Lemke-Howson algorithm as presented in class was formulated for finding symmetric equilibria in 2-player symmetric games. In this problem, you will prove that this is, in fact, without loss of generality.

   Let NASH be the problem of, given any game, finding a Nash equilibrium.

   Define a symmetric game as a game in which changing the identity of players does not change the game from their point of view. Formally, in the special case of two-player games, a symmetric game is one in which both players have the same set of strategies and the matrix of payoffs to player one is the transpose of the matrix of payoffs to player two (Lecture 8, Slide 16). Furthermore, define SYMMETRIC NASH to be the problem of, given a symmetric game, finding a symmetric Nash equilibrium (i.e., a Nash equilibrium in which all players use the same (possibly mixed) strategy).

Prove that there exists a polynomial-time reduction from NASH to SYMMETRIC NASH for two-player games. That is, given a black box solution for SYMMETRIC NASH, prove that there exists a polynomial-time algorithm that, on any given two-player game, uses a single query to the black box and outputs a solution to NASH.

4. **Equilibrium Computation in Anonymous Games** (25 points)

   Define an *anonymous game* as a game in which each player's utility does not depend on the identity of the other players. Formally, an anonymous game G consists of a set N of n players, a set S of m strategies available to all players, and a set of utility functions u^i_ℓ, each of which represents the payoff to agent i for playing strategy ℓ, where the payoff to each player only depends on her choice of action and the number of other players who chose each strategy. Furthermore, throughout this problem, assume that all utilities are normalized to take values in $[0,1]$.

   Now, define the *total variation distance* between two distributions P and Q supported on a finite set A as

$$\|P-Q\|_{TV} = \frac{1}{2}\sum_{\alpha\in A}|P(\alpha)-Q(\alpha)|.$$

   For anonymous games where the strategy space is of constant size, it is possible to show that there exists a PTAS (polynomial-time approximation scheme) for mixed Nash equilibrium. The idea is similar to the Lipton-Markakis-Metha algorithm we saw in class: show that there is a relatively small set of candidates ("discretized" strategy profiles), and then check them one-by-one through enumeration. The computational component is a bit tedious in this case, so this problem focuses on the existence component. Moreover, to make things simple, assume that there are two strategies per player, which means that we can represent a mixed strategy profile as a vector of probabilities $(q_1,\dots,q_n)$, where q_i is the probability that player i plays strategy 1.

   We will use the following (highly nontrivial) theorem. Intuitively, it asserts that any arbitrary sequence of probabilities can be well-approximated by another sequence of discretized probabilities in a way that preserves certain desirable properties.

   Theorem 1. *Let $p_1,\dots,p_n$ be n arbitrary probabilities, that is, $p_i\in[0,1]$ for $i=1,\dots,n$. Furthermore, let $X_1,\dots,X_n$ be n independent indicator random variables such that $\mathbb{E}[X_i]=p_i$ for all i. Finally, let k be a positive integer. Then there exists another set of probabilities $q_1,\dots,q_n$ that satisfy the following properties.*

   - $|q_i-p_i| = O(1/k)$ *for all* $i\in[n]$*;*
   - q_i *is an integer multiple of* $1/k$ *for all* $i\in[n]$*;*
   - *given independent indicator random variables $Y_1,\dots,Y_n$ such that $\mathbb{E}[Y_i]=q_i$ for all i, we have that for all $j=1,\dots,n$,*

$$\left\|\sum_{i\neq j}X_i-\sum_{i\neq j}Y_i\right\|_{TV} = O\left(\frac{1}{\sqrt{k}}\right);$$

- *for all i, if $p_i = 0$, then $q_i = 0$ and if $p_i = 1$, then $q_i = 1$ (i.e., for all i, the support of Y_i is contained in the support of X_i).*

Note that, say, $\sum_{i=1}^{n} X_i$ can naturally be interpreted as a probability distribution over $\{0, 1, \ldots, n\}$.

Using the theorem, show that, for every two-strategy anonymous game with n players and all $k \in \mathbb{N}$, there exists a mixed strategy profile $(q_1, \ldots, q_n)$, where each q_i is a multiple of $1/k$, that constitutes an $O(1/\sqrt{k})$-approximate mixed Nash equilibrium for the game; i.e., no player can gain more than $O(1/\sqrt{k})$ by deviating.