CMU 15-896

Fair division: Cake cutting algorithms

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Cake cutting

- A cake must be divided between several children
- The cake is heterogeneous
- Each child has different value for same piece of cake
- How can we divide the cake fairly?
- What is "fairly"?
- A metaphor for land disputes, time using shared resources, etc.

The model

- Cake is interval [0,1]
- Set of agents/players $\mathrm{N} = \{1, \ldots, n\}$
- Piece of cake $X \subseteq [0,1]$: finite union of disjoint intervals
- Each agent has valuation V_i over pieces of cake
  - Additive: for $X \cap Y = \emptyset$, $V_i(X) + V_i(Y) = V_i(X \cup Y)$
  - For all $i \in N$, $V_i([0,1]) = 1$
  - Divisible: $\forall \lambda \in [0,1]$ can cut $I' \subseteq I$ s.t. $V_i(I') = \lambda V_i(I)$
- Find allocation $A_1, \ldots, A_n$
  - Not necessarily connected pieces

Fairness properties

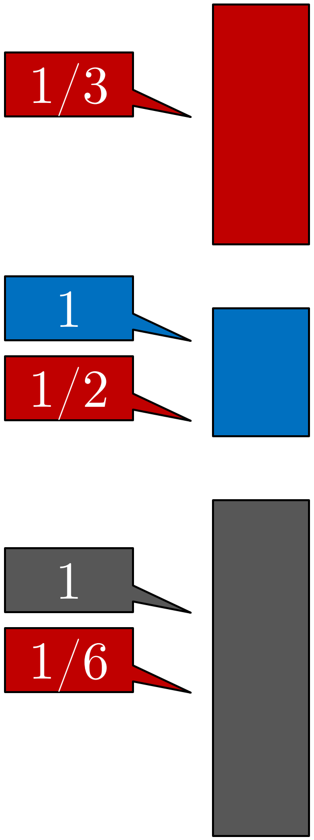
- Proportionality:

$$\forall i \in N, V_i(A_i) \geq \frac{1}{n}$$

- Envy-Freeness (EF):

$$\forall i, j \in N, V_i(A_i) \geq V_i(A_j)$$

- Vote: For n = 2 which is stronger?
- Vote: For n ≥ 3 which is stronger?

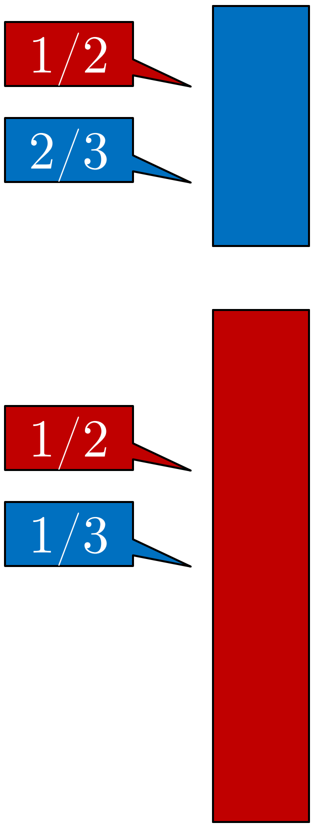


Cut-and-Choose

- Algorithm for $n = 2$
- Agent 1 divides into two pieces X, Y s.t.

$$V_1(X) = 1/2 , V_1(Y) = 1/2$$

- Agent 2 chooses preferred piece
- This is EF (hence proportional)



The Robertson-Webb model

- A concrete complexity model
- Two types of queries
    - $\text{Eval}_i(x, y) = V_i([x, y])$
    - $\text{Cut}_i(x, \alpha) = y$ s.t. $V_i([x, y]) = \alpha$
- Vote: Minimum #queries needed to find an EF allocation when $n = 2$?

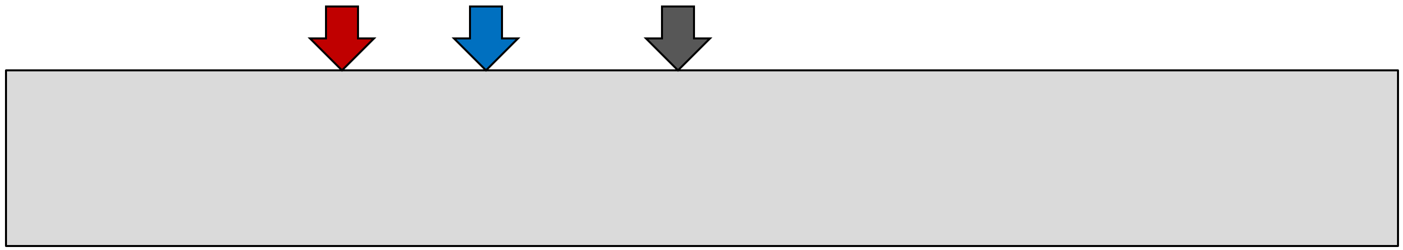
Dubins-Spanier

- Referee continuously moves knife
- Repeat: when piece left of knife is worth $1/n$ to agent, agent shouts “stop” and gets piece
- That agent is removed
- Last agent gets remaining piece
- Protocol is proportional

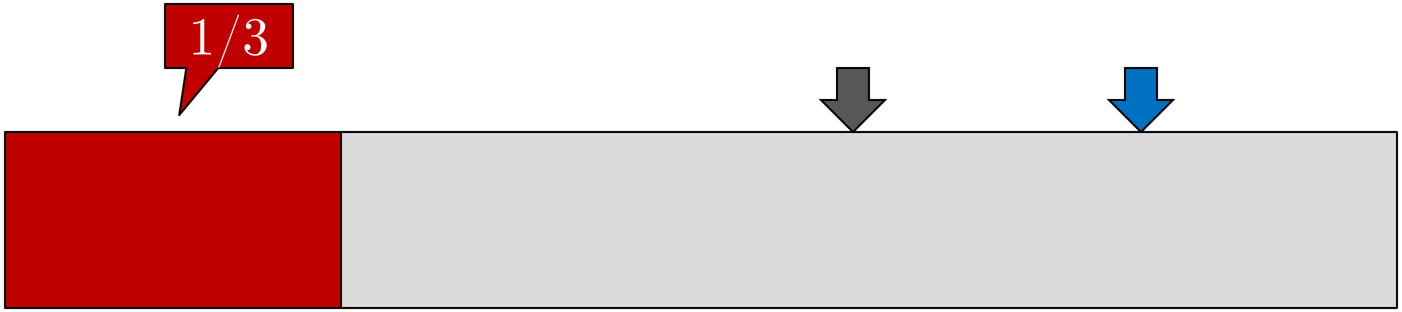
Discrete Dubins-Spanier

- Moving knife is not really needed
- Repeat: each agent makes a mark at his $1/n$ point, leftmost agent gets piece up to its mark
- The protocol is proportional

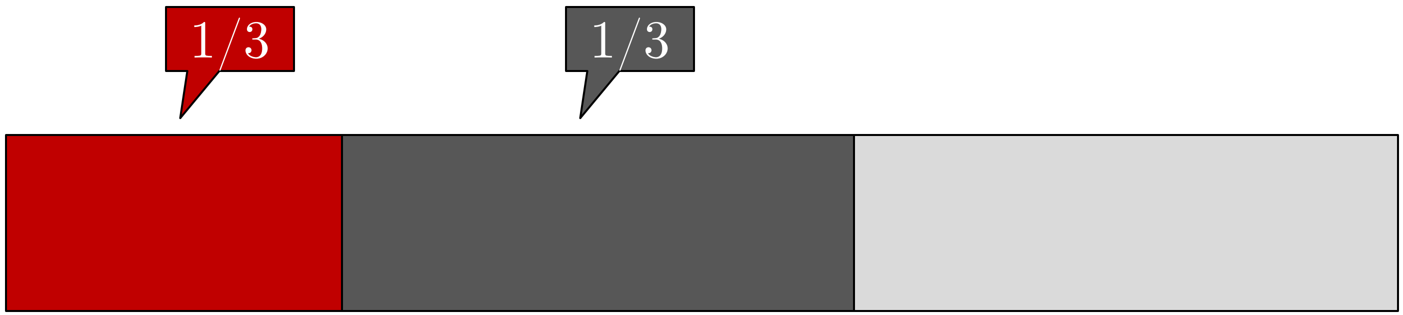
Example



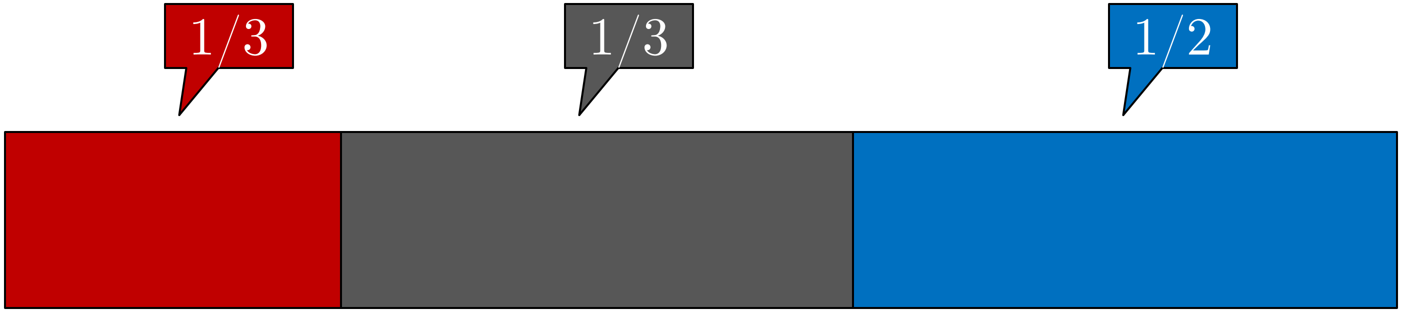
Example



Example



Example



Even-Paz

- Given $[x, y]$, assume $n = 2^k$
- Each agent i makes a mark z such that
$$V_i([x,z]) = \frac{1}{2} V_i([x,y])$$
- Let z^* be the $n/2$ mark from the left
- Recurse on $[x, z^*]$ with the left $n/2$ agents, and on $[z^*, y]$ with the right $n/2$ agents
- The protocol is proportional

Complexity of proportionality

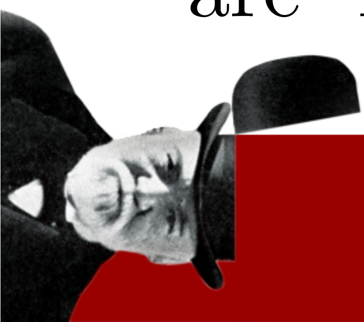
- Dubins-Spanier requires $\Theta(n^2)$ queries in the RW model
- Even-Paz requires $\Theta(n \log n)$ queries in the RW model
- **Theorem** [Edmonds and Pruhs, 2006]: Any proportional protocol needs $\Omega(n \log n)$ [We'll prove on Tuesday]

Selfridge-Conway

- **Stage 0**
  - Agent 1 divides the cake into three equal pieces according to V_1
  - Agent 2 trims the largest piece s.t. there is a tie between the two largest pieces according to V_2
  - Cake 1 = cake w/o trimmings, Cake 2 = trimmings
- **Stage 1 (division of Cake 1)**
  - Agent 3 chooses one of the three pieces of Cake 1
  - If agent 3 did not choose the trimmed piece, agent 2 is allocated the trimmed piece
  - Otherwise, agent 2 chooses one of the two remaining pieces
  - Agent 1 gets the remaining piece
  - Denote the agent $i \in \{2, 3\}$ that received the trimmed piece by T, and the other by T'
- **Stage 2 (division of Cake 2)**
  - T' divides Cake 2 into three equal pieces according to $V_{T'}$
  - Agents T, 1, and T' choose the pieces of Cake 2, in that order

RW IS FOR HONEST KIDS

- EF protocol that uses n queries
- f = 1-1 mapping from valuation functions to [0,1]
- The protocol asks each agent $\mathsf{cut}_i(0, 1/2)$
- Agent i replies with $y_i = f(V_i)$
- The protocol computes $V_i = f^{-1}(y_i)$
- We therefore need to assume that agents are "honest"



Complexity of EF

- $n = 2$: Cut and Choose
- $n = 3$: "good" protocol [Selfridge and Conway]
- $n \geq 4$: known protocol requires unbounded #queries [Brams and Taylor, 1995]
- Lower bound of $\Omega(n^2)$ [P, 2009], unbounded with contiguous pieces [Stromquist, 2009]

Price of fairness

- Social welfare of $A = \sum_{i \in N} V_i(A_i)$
- Requires interpersonal comparison of utils
- Price of EF = worst-case (over valuation functions) ratio between social welfare of the best allocation and social welfare of the best EF allocation
- **Theorem** [Caragiannis et al. 2009]: The price of EF is $\Omega(\sqrt{n})$

Proof of Theorem

- Agents $1, \dots, \sqrt{n}$ uniformly desire disjoint intervals of length $1/\sqrt{n}$
- The others uniformly desire the whole cake
- Optimal solution: give whole cake to the "focused" agent $\Rightarrow \mathrm{SW} = \sqrt{n}$
- Any EF solution must give $\frac{n-\sqrt{n}}{n}$-fraction to the "unfocused" agents $\Rightarrow \mathrm{SW} \leq 2$ ■

The dumping paradox

- If connected pieces must be allocated, by throwing away pieces, can increase the welfare of optimal EF allocation by a factor of $\sqrt{n}$ [Arzi et al. 2011]
- Example: for $n = 2$, can increase from 1 to $\sim 3/2$