Algorithms, Games, and Networks	February 7, 2013
Lecture 8	
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1 Overview

In this lecture, we discuss the topic of social choice by exploring voting rules, axioms, and applications. The following items are covered:

- Brief History of Social Choice Theory
- Definition of Voting Model
- Voting Rules
- Axioms of Social Choice
- Practical Issues and Applications

The relevant reading is Section 9.2 of the AGT book.

2 Social Choice Theory

Social choice theory is a mathematical theory that deals with aggregation of individual preferences. For example, the Presidential Election is one extremely simple example of social choice where people vote based on their preferences over candidates. This area was essentially founded by the 18th Century mathematicians, Condorcet and Borda. After the contribution of Charles Dodgson from the 19th Century, social choice theory as we know it today has begun in the 20th Century with the work of the Nobel prize winner, Kenneth Arrow.

3 Voting Model

The voting model is a traditional abstraction used in social choice theory:

- There is a set of n voters, $N = \{1, ..., n\}$.

- There is a set of m alternatives A where $|A| = m$.
- Each voter has a ranking over the alternatives
- $x \succ_i y$ means that voter i prefers x to y
- The preference profile is a collection of all voters' rankings

For example, the following table represents the preference profile of 5 people about international cuisine.

1	2	3	4	5
In	In	C	J	C
It	C	J	In	J
J	A	In	C	It
C	It	It	It	A
A	J	A	A	In

Types of Cuisine: Indian (In), Chinese (C), Japanese (J), American (A), Italian (It)

Person 1 prefers Indian to Italian to Japanese to Chinese to American. Note that this preference profile will be used throughout the lecture.

4 Voting Rules

A voting rule is defined as a function from preference profiles to alternatives that specifies the winner of the election. Here we introduce several famous voting rules:

- Plurality: Each voter awards one point to his/her top alternative, and the alternative with the most points wins. This is the one used in almost all political elections. In the cuisine example, both Indian cuisine and Chinese cuisine obtain 2 points. To determine only one winner, we need to specify what to do in the case of ties.
- Borda Count: Proposed by the 18th Century mathematician, chevalier de Borda, each voter awards $m - k$ points to his/her k'th ranked alternative. Same as Plurality, the alternative with the most points wins. This is used for elections to the national assembly of Slovenia and the Eurovision song contest. In the cuisine example, Chinese cuisine wins with 14 points.
- Veto: Each voter decides his/her least preferred alternative, and the alternative with the least vetoes wins. In the cuisine example, Italian cuisine and Chinese cuisine are tied with 0 vetos.

The above voting rules are categorized as positional scoring rules since they can be defined by a vector $(s_1, ..., s_m)$ where each voter gives s_k points to k'th position. Plurality is defined by $(1, 0, ..., 0)$, Borda Count is defined by $(m-1, m-2, ..., 0)$, and Veto is defined by $(1, ..., 1, 0)$.

Some voting rules have multiple rounds to determine the winner. We adopt a new concept called a pairwise election; alternative x is said to beat alternative y in a pairwise election if the majority of voters prefer x to y. Here are the voting rules with more than one round:

- Plurality with Runoff: In the first round, two alternatives with highest plurality scores survive. In the second round, the winner of a pairwise election between those two alternatives becomes the final winner. In the cuisine example, Indian cuisine and Chinese cuisine survive after the first round, and Indian cuisine becomes the final winner by winning a pairwise election against Chinese cuisine.
- Single Transferable Vote (STV): There are $m-1$ rounds. In each round, the alternative with the least plurality votes is eliminated, and the alternative survived to the last becomes the winner. This is used in Ireland, Malta, Australia, and New Zealand (and Cambridge, MA). In the cuisine example, American, Italian, Japanese, and Chinese cuisine will be removed in that order, and Indian cuisine will be the final winner.

5 Social Choice Axioms

The desirable properties of voting rules should be defined to choose among the different voting rules for different situations. We are going to define two social choice axioms, Majority Consistency and Condorcet Consistency as such desirable properties.

5.1 Majority Consistency (MC)

Given a voting rule that satisfies Majority Consistency, if a majority of voters rank alternative x first, then x should be the final winner. Note that Plurality satisfies Majority Consistency by definition while Borda Count does not. A counterexample for Borda Count is following:

1	2	3
a	a	c
c	c	b
b	d	d
d	b	a

Although a majority of voters rank a first, c is the winner of Borda Count.

5.2 Condorcet Consistency (CC)

Marquis De Condorcet was a French mathematician, philosopher, and political scientist in the 18th Century. He introduced the ideas of Condorcet Winner and Condorcet Paradox in his old writings. Recall that alternative x beats alternative y in a pairwise election if a majority of voters rank x above y. Condorcet Winner beats every other alternative in pairwise election, and Condorcet Paradox points out the fact that there could be a cycle in majority preferences so that there is no Condorcet Winner. An example preference profile that demonstrates the Condorcet Paradox is the following:

1	2	3
a	c	b
b	a	c
c	b	a

Note that a beats b, b beats c, and c beats a in pairwise election. Therefore, there's no Condorcet Winner in the above situation. A Condorcet-consistency voting rule should select a Condorcet Winner as the final winner if one exists. For instance, neither Borda Count nor STV satisfy Condorcet Consistency. A counterexample for Borda Count is the same one used to prove Borda Count does not satisfy MC. There, a is a Condorcet Winner, but c is the winner under Borda Count. A counterexample for STV is following:

1	2	3
b	c	d
a	a	a
c	d	b
d	b	c

Although a is Condorcet Winner, it will be eliminated in the first round.

The following voting rules do satisfy Condorcet Consistency:

- Copeland: The score of an alternative is the number of alternatives it beats in pairwise elections, and the alternative with the highest score wins. This voting rule naturally satisfies Condorcet Consistency; if a Condorcet Winner exists, it will score $m - 1$ which is the highest possible score.
- Maximin: The score of alternative x is defined as $min_y|i \in N : x \succ_i y|$, and the alternative with the highest score wins. This voting rule satisfies Condorcet Consistency; if a Condorcet Winner exists, its score is more than $n/2$, and and the score of every other alternative is less than $n/2$.
- Dodgson's Rule: A distance function between preference profiles is defined as the number of swaps between adjacent candidates, and the Dodgson score of alternative

x is the minimum distance from a profile in which x is a Condorcet Winner. Dodgson's Rule selects the candidate that minimizes Dodgson score. The problem of computing Dodgson score is NP-complete. In the following example, a needs to beat b and e to become a Condorcet Winner, and therefore, the Dodgson score of a is (1 swap in voter 2) + (2 swaps in voter 5) = 3.

1	2	3	4	5
a	b	e	e	b
b	a	b	c	e
c	c	c	d	d
d	d	a	a	a
e	e	d	b	c

5.3 Relation between MC and CC

An alternative ranked first by a majority of voters is always a Condorcet Winner. Therefore, if a majority of voters rank alternative x first, a voting rule that satisfies CC always selects x as the final winner because it is Condorcet Winner. It means CC implies MC. However, there's no guarantee that a voting rule with MC will always select a Condorcet Winner if one exists. Consider the counterexample used to show STV doesn't satisfy CC. Although a is a Condorcet Winner, Plurality which satisfies MC will select one among b, c, and d as a winner. Consequently, MC does not imply CC. In conclusion, CC $\implies$ MC.

6 Practical Issues and Applications

6.1 Awesome Example

33 voters	16 voters	3 voters	8 voters	18 voters	22 voters
a	b	c	c	d	e
b	d	d	e	e	c
c	c	b	b	c	b
d	e	a	d	b	d
e	a	e	a	a	a

In the above example, Plurality selects a, Borda Count selects b, c is a Condorcet winner, STV selects d, and Plurality with Runoff selects e. This is a good demonstration of the fact that different voting rules may result in different winners.

6.2 Social Choice in Real World

Despite the academic agreement that STV is better, the UK referendum to change the method for electing Members of Parliament from Plurality to STV was rejected because of short-term political reasons. This is one example of practical barriers against applications social choice theory.

However, the theory of computational social choice is applicable to human computation systems and multiagent systems where the system designers have freedom to employ any voting rule they want for maximizing positive results through computational thinking.

6.3 Applications

- Web Search: Generalized Condorcet is defined when there is a partition X and Y of A such that a majority of voters prefer every $x \in X$ to every $y \in Y$. Then, the rule ranks X above Y. Assuming spam websites are identified by a majority of search engines, we can make rank these websites last by aggregating results from different search engines.
- Robobees: Robobees need to decide on a joint plan (alternative) out of many possible plans. Each Robobee (agent) has a numerical evaluation (utility) for each alternative, and we want to maximize the sum of all the utilities (social welfare). Using Plurality may select a bad alternative when $n/2-1$ agents have utility only for a while $n/2+1$ agents have utility for both all alternatives such that the utility of b is a little greater than the utility of the other alternatives. In this case, Plurality will choose b although a maximizes social welfare. To avoid this problem, we can make each agent votes for an alternative with probability proportional to its utility. It has been proven this approach gives almost optimal social welfare in expectation under proper conditions.