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| **Algorithms, Games, and Networks** | April 2, 2013 |
| Lecture 20 | |
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Kidney failure is a big problem, obviously. Dialysis is not a great option, transplants are better. The number of kidneys in the donor kidney supply is limited and the demand remains much higher. The best option is transplantation from a live donor but there are many issues of compatibility.

The notion of Kidney exchange is to match incompatible donor-patient pairs with other incompatible pairs.

More generally, we model with a directed graph $G = (V, E)$, each $v$ in the set $V$ is a donor-patient pair. Edge $(u, v)$ exists in $E$ if the donor in $u$ is compatible with the patient in $v$. We then exchange along cycles, optimally, pairwise cycles.

# 1 Cycle Cover

Cycle cover aims for maximum cover by cycles. If a cycle is unrestricted then the problem is in P. But, cycle cap is a medical necessity, because, transplants must be done simultaneously as there is no way to legally enforce the exchange of body parts. Cycles of length K require 2K operating rooms (one patient and each donor) which makes the logistics very hard, so 3 cycles tends to be the upper bound.

**Theorem 1 (Abraham et al. 2007)** *Given $G$, $L \leq 3$, computing a max cycle cover with cycles of length $\leq L$ is NP-hard.*

This is a somewhat unintuitive claim, because for $L = 2$ the problem is is simply the maximum cardinality matching problem in an undirected graph, which can be solved efficiently. For a proof of the Theorem, see Theorem 1 of Abraham et al. [1].

## 1.1 Cycle cover in practice

The projected size of the US national kidney exchange pool is 10,000, but even at that scale we can still compute this match in a reasonable amount of time. (see slide 7 for more details)

## 2 Are long cycles needed?

A donor is compatible with a patient if the patient has 'more letters' in their blood type (O is empty set). Example - A can donate to A or AB but not B or O. (the donors letters need to be a subset of the donors)

In the following case we assume there are no issue type incompatibilities between pairs but this is not strictly the case in practice. As there are two types of incompatibilities: blood type and tissue type.

### 2.1 Classification of Pairs

We classify donor-patient pairs into 4 types.

- Self-demanded: X-X
- Reciprocally-demanded: A-B and B-A
- Over-demanded: X-Y that are blood-type compatible
- Under-demanded: X-Y that re blood-type incompatible

Based on this classification there are 16 possible vertex types (see slide 12 for illustration). We assume that there is an infinite supply of under-demanded pairs.

### 2.2 The structure of the optimal 2 cycle matching:

First, the over demanded can match with under demanded (assuming the pools are infinite sized). Because each over-demanded vertex can only help one vertex, and because there are unlimited pairs, this is optimal.

Then, reciprocal demands could be matched with over demanded but can't because under demanded has taken them all. So we match them internally. Self demanded are also not possible to match externally because all others are taken, so we match as many as possible matches among same pair types. (illustration on slide 12)

### 2.3 The structure of the optimal 3 cycle matching:

With 3 we can deal with odd numbers of self demanded pairs.Also, AB-O pairs can form 3 cycle chains with O-A, A-AB or O-B, B-AB, and remaining reciprocal demanded can be matched in 3 cycles with A-O and B-O (together). So we get more exchanges.

Assuming that we draw each of the $n$ pairs from product distribution over blood types; each type has a constant probability, which of these three opportunities gives $\Omega(n)$ extra matches?

It turns out matching with AB-O pairs offers the most new matches because it helps 2 under demanded donors which will lead to extra match for each AB-O pair. Matching with self demanded pairs only adds a small number of new matches, while A-O and B-O matching adds a reasonable number but the influence depends on the number of "leftover" A-B and B-A pairs, which is $o(n)$.

## 3 A random graph model

Each blood type X has probability $\mu_X$. Draw blood types for patient and donor. Blood type comparable donor and patients are tissue incompatibility with $\gamma > 0$.

If donor patient pair is internally compatible, remove them, otherwise, randomly generate edges to blood type comparable pairs.

Tissue type compatibility probabilities are critical to finding good matches.

**Theorem 2** *[3] In large random graphs you don't need cycles with $L \leq 3$*

## 4 Chains

Altruistic donors can initiate a chain. Sometimes people offer a kidney for free to whomever needs it. This can lead to long change that are not simultaneous and has a huge impact on overall matches made.

See slide 17 for a chart on cardinality with longer chains.

In large exchanges, chains provide less substantial benefits

**Theorem 3** *[4, 2] In large random graphs w.h.p $\exists$opt allocation with cycles of length $\leq 3$.*

Basically, you don't gain from longer chains. (see slide 18 for evidence of this theorem)

## 5 Introducing Cross-matches

Cross-matching involves mixing cells and serum to determine whether a patient will reject the kidney. Blood samples must be in physical contact to do this, making it expensive and logistically difficult.

It adds another level of uncertainty and we assume that cross-match is negative with some probability, so optimization should favor short cycles and short chains. Although this makes matching harder, in practice, it works quite well. (see slide 20 for real world results and 21 for simulated ones)

## 6 Introducing Dynamics

People are always entering and leaving the pool, this makes the problem dynamic and means that the long term view may be more important. Matching myopically is not always optimal as people die, so we tend to elect not to save great pairs for the future.

A partial solution is to assign a weight to each vertex that shows its potential to have more value in the future. In each round, we maximize cardinality of matching with the total potential removed. We optimize potential using a local search, essentially running a simulation many times with different parameters to work out what works best.

## References


[1] D. J. Abraham, A. Blum, and T. Sandholm. Clearing algorithms for barter exchange markets: Enabling nationwide kidney exchanges. In *Proceedings of the 8th ACM Conference on Electronic Commerce (EC)*, pages 295–304, 2007.

[2] I. Ashlagi, D. Gamarnik, M. A. Rees, and A. E. Roth. The need for (long) chains in kidney exchange. Manuscript, 2011.

[3] I. Ashlagi and A. Roth. Individual rationality and participation in large scale, multi-hospital kidney exchange. In *Proceedings of the 12th ACM Conference on Electronic Commerce (EC)*, pages 321–322, 2011.

[4] J. P. Dickerson, A. D. Procaccia, and T. Sandholm. Optimizing kidney exchange with transplant chains: Theory and reality. In *Proceedings of the 11th International Joint Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 711–718, 2012.