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Lecture 17

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1 Setting and Model

Suppose we have a cake that is to be divided among several people and that people may disagree on the value of portions of the cake. For example, a piece worth half the cake to one person may be worth only a quarter to another. We examine the question of how to divide such a cake among the people so that we are "fair" in some sense.

To begin, we first rigorously define the model. The cake itself is represented by the real interval $[0, 1]$ and a piece of this cake is a finite union of disjoint intervals $X \subseteq [0, 1]$. Furthermore, each agent i has a valuation function V_i that represents his/her assessment of pieces of the cake. That is, $V_i(X) = x$ states that the piece X has value x to i. We will assume such V_i have the following properties:

- $V_i([0, 1]) = 1$
- $V_i([x, x]) = 0$ for every $x \in [0, 1]$
- if $X \cup Y = \emptyset$ then $V_i(X) + V_i(Y) = V_i(X \cup Y)$

An allocation of the cake among n agents is then a partition of $[0, 1]$ call $A_1, A_2, ..., A_n$, such that agent i receives A_i.

The question we concern ourselves with is to find allocations with certain desirable properties. Two such important properties are the following:

- Proportionality: $\forall i; V_i(A_i) \geq 1/n$
- Envy-Freeness (EF): $\forall i, j; V_i(A_i) \geq V_i(A_j)$

Intuitively, proportionality implies that each agent believes he/she has a fair share of the cake, and envy-freeness implies he/she has no incentive to trade their own piece for another's. Clearly, in the case of two agents these two properties are equivalent, and for any higher number of agents, envy-freeness is a strictly stronger property than proportionality.

Before we begin and due to the continuous (as opposed to discrete) nature of the valuation functions, we must establish a model to interact with said valuation functions. Perhaps the most popular approach is the Robertson-Webb model. This model only allows two operations in order to interact with the valuation functions - which are:

1. Ask an agent what an interval is worth to them. That is, for any agent i and interval $[a, b]$ query for $V_i([a, b])$.
2. Ask an agent to produce an interval whose left endpoint is given and of certain worth to them - or report impossibility. That is for any agent i, and $a, v \in [0, 1]$ query for a b such that $V_i([a, b]) = v$ or report no such b exists.

Generally such models are not critical for our understanding, however they are useful regarding complexity results. For example, we will see later that proportionality requires $\Omega(n \lg n)$ queries and in a recent paper, it was proved that envy-freeness requires $\Omega(n^2)$ queries. Importantly, such complexity results assume the agents respond truthfully to the queries as otherwise they can encode their entire valuation function into the response of a single cut/evaluation query (assuming some very non-restrictive assumptions on the valuation functions).

2 Famous Cake Cutting Protocols

In this section we give a few famous protocols.

2.1 Cut-and-Choose

This protocol is only for two agents and produces an EF allocation. The first agent is told to cut the cake into two pieces - which are equal in value to him/her - and the other gets to choose which of the two pieces to take. The remaining piece is given to the first agent.

2.2 Dubins-Spanier

This protocol produces a proportional allocation for any number of agents with $\Theta(n^2)$ queries. The classic formulation of the protocol involves a continuous process, but the discrete process is more elegant and simple, and so we discuss that here. The protocol is defined by n rounds in which a piece is allocated to a agent in every round. In the first round, all agents are asked to make a mark on the cake such that the left end to their mark represents $1/n$ of the cake's worth. An agent with the leftmost mark is then given that piece of the cake. We then repeat the process for the remaining $n - 1$ agents in the second round. This continues until a single agent is left who receives the remaining part of the cake in the n^{th} and final round.

2.3 Even-Paz

This protocol produces a proportional allocation for any number of agents with $\Theta(n \lg n)$ queries. The protocol is similar to Dubins-Spanier but uses a binary search like approach to reduce the complexity (i.e. number of queries). In the beginning we ask all n agents to make a mark on the cake such that the left end to their mark represents $\lfloor n/2 \rfloor$ of the cake. The creators of the $\lfloor n/2 \rfloor$ leftmost marks are then given the section of the cake from the left end to the rightmost marks of the $\lfloor n/2 \rfloor$ leftmost marks, and the remaining agents the remaining part of the cake. This process is recursively done to allocate the cake.

The $\Theta(n \lg n)$ queries is notable as next class we will see that proportionality requires $\Omega(n \lg n)$ queries and so this is optimal.

2.4 Selfridge-Conway

This protocol produces an EF allocation for three agents.

- Stage 0:
  - Agent 1 divides the cake into three equal pieces according to himself.
  - Agent 2 trims the largest piece s.t. there is a tie between the two largest pieces according to himself.
  - Let Cake 1 be the cake without the trimmings, and Cake 2 the trimmings.
- Stage 1:
  - Agent 3 chooses one of the three pieces of Cake 1.
  - If agent 3 did not choose the trimmed piece, agent 2 is allocated the trimmed piece.
  - Otherwise, agent 2 chooses one of the remaining two pieces.
  - Agent 1 gets the remaining piece.
  - Denote the agent $i \in \{2, 3\}$ that received the trimmed piece by T and the other by T'.
- Stage 2:
  - T' divides Cake 2 into three equal pieces according to himself.
  - Agents $T, 1$, and T' choose the pieces of Cake 2, in that order.

To see why this produces an EF allocation, consider first the allocation of just Cake 1 (completed in Stage 1). Certainly agent 3 will be envy-free as he chooses his piece first. Agent 2 will always receive one of the pieces he deemed tied for largest and so he cannot be envious either. Finally, agent 1 is envy-free as the two untrimmed pieces are each of worth 1/3 value to him - one of which he receives (since the untrimmed piece must be taken by agent 2 if agent 3 did not opt for it). Now consider Cake 2. As T' gets to choose first he cannot be envious of anyone. Similarly, Agent 1 cannot be envious of T. However, it is possible he deems the piece of cake 2 chosen by T' to be of higher worth than his own piece. Fortunately, as the entire Cake 2 in addition to the piece from Cake 1 allocated to agent T' is worth the same as agent 1's piece from cake 1 (to agent 1) he cannot be envious of agent T'. Finally, T cannot be envious as he cuts the pieces so that they are of equal size.

The question of proportionality was largely answered by the Even-Paz protocol, but the question of EF allocations appears to be much harder. This protocol solved the conundrum for 3 agents, but for any more agents there currently exists no algorithm that runs in bounded time (and it is strongly suspected no such algorithm exists). However, the Brams-Taylor protocol - which is not discussed here - does compute EF allocations in a finite number of queries, but requires an unbounded number of queries in the number of players. That is, the protocol will always terminate with a EF allocation, but may take arbitrarily long (depending on the valuation functions).

3 Price of Fairness

Although the inherent fairness of EF allocations is a desirable property, the social welfare may suffer greatly due to it. Specifically, if the social welfare is defined to be the sum of everyone's valuation of their own piece, the ratio of the best allocation (in terms of social welfare) to the best EF allocation can be in $\Omega(\sqrt{n})$.

The proof proceeds as follows. Suppose agents $1, 2, ..., \sqrt{n}$ uniformly desire disjoint intervals of length $1/\sqrt{n}$ and all other agents uniformly desire the whole cake. The best allocation for social welfare will give the first $\sqrt{n}$ agents their entire desired interval. This gives a social welfare of at least $\sqrt{n}$. Unfortunately, in any EF allocation, we must ensure that at least $(n - \sqrt{n})/n$ of the length of cake go to agents who value the cake uniformly. In total such agents can clearly at most add 1 to the social welfare. Moreover, as the first $\sqrt{n}$ agents receive a section of cake of length at most $\sqrt{n}/n = 1/\sqrt{n}$ they can also produce at most 1 for the social welfare. Thus, any EF allocation has a social welfare of at most 2. This gives the result.

Interestingly, at times the option to throw away parts of the cake can increase social welfare. In 2011 it was proven that if contiguous pieces must be allocated, removal of pieces can increase the social welfare of the optimal EF allocation by a factor of $\sqrt{n}$.