**Algorithms, Games, and Networks** February 21, 2013

Lecture 12

*Lecturer: Ariel Procaccia* *Scribe: Sercan Yıldız*

# 1 Overview

In this lecture, we introduce the axiomatic approach to social choice theory. In particular, we focus on the example of PageRank and study a representation theorem due to Altman and Tennenholtz. We also consider the impartial $k$-selection problem and see how randomization allows us to design an algorithm with a finite approximation guarantee for this problem. The latter is a result of Alon et al.

# 2 Axiomatic Approach

The axiomatic approach to social choice aims to characterize a (family of) voting rule(s) as the only one satisfying a set of axioms. For instance, a classical result of Young shows that a voting rule is a positional scoring rule if and only if it satisfies anonymity, neutrality and consistency. In this context, we say that a voting rule is *anonymous* if its output does not depend on the identities of the voters. More precisely, the linear ordering produced by the voting rule is invariant under permutation of the voter names. Similarly, a voting rule is *neutral* if its output does not depend on the identities of the candidates. Finally, a voting rule is *consistent* if whenever we split the set of voters into two groups and an alternative wins with respect to both, it also wins with respect to the full preference profile. Results of this kind are called representation theorems in the literature. This approach has been applied to various domains such as ranking systems, collaborative filtering and recommendation systems. We study a representation theorem for PageRank, the algorithm used by Google to rank web pages.

## 2.1 PageRank

We are going to look at a simplified version of PageRank which captures the essence of the full algorithm. Suppose we represent the internet with a directed graph $G = (V, E)$ in which each vertex is a web page and an edge $(x, y) \in E$ corresponds to a hyperlink from page $x$ to page $y$. Given a directed graph $G$ as input, a ranking system produces a ranking over $V$ that represents the "power" or "relevance" of web pages. From a social choice perspective, this has two major differences from the setting that we have seen in the previous lectures.

The first is that the set of voters and the set of alternatives coincide. Second, instead of expressing their preferences by ranking the alternatives, the voters express their preferences through votes of support, or outgoing hyperlinks. An intuition that people often use to think about PageRank is the "random surfer" model. In this model, the surfer starts at some web page, picks a hyperlink on that page uniformly at random and follows it to the next page and so on.

For technical reasons, we assume that the graph is strongly connected. Let $S(v_j)$ be the set of successors of vertex $v_j$ in $G$. Let $A_G$ be the $n$-by-$n$ matrix whose entry at position $(i, j)$ is $\frac{1}{|S(v_j)|}$ if $(v_j, v_i) \in E$ and 0 otherwise. Given that the surfer is currently at vertex $v_j$, $[A_G]_{ij}$ is exactly the conditional probability that the surfer will be at vertex $v_i$ next. Because the graph is strongly connected, this distribution will converge to a stationary distribution $r$ which satisfies

$$A_G r = r. \tag{1}$$

PageRank ranks the vertices based on this stationary distribution. In particular, it lets

$$v_i \succeq_{PR} v_j \Leftrightarrow r_i \geq r_j. \tag{2}$$

Note that PageRank allows pages to have the same rank.

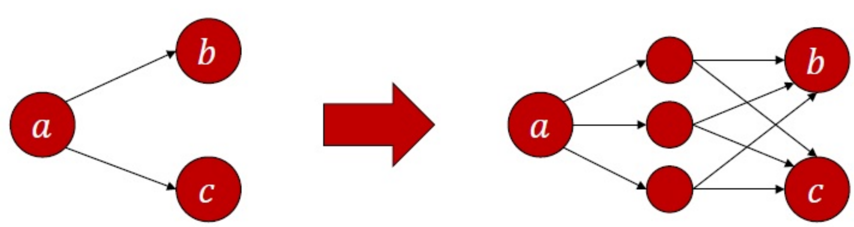
Next, we give five axioms which are satisfied by PageRank.

### Axiom 1: Isomorphism

A ranking system satisfies *isomorphism* if its ranking does not depend on the names of the vertices, only on the voting structure. If two vertices $a$ and $b$ are symmetric in the sense that they have the same sets of predecessors and successors, this axiom requires that their rankings are the same. It is not difficult to see that this requirement is satisfied by PageRank because vertices $a$ and $b$ will have the same stationary probability under these circumstances.

### Axiom 2: Vote by Committee

A ranking system satisfies *vote by committee* if a vertex can indirectly vote through a committee which casts the same set of votes as it does. In the example below, each member of the committee has the same set of outgoing edges as vertex $a$, namely, one edge going to vertex $b$ and one edge going to vertex $c$.



The axiom requires that if we "replace" a vertex with a committee, then the ranking over the original set of vertices stays the same. Note that this transformation does not change the relative probabilities of the original vertices in PageRank. More formally, we say that a ranking system $f$ satisfies vote by committee if, for all $G = (V, E)$, all $v, v', v'' \in V$ and all $k \in \mathbb{N}$,

$$[v' \succeq_G^f v''] \Leftrightarrow [v' \succeq_{G'}^f v''] \tag{3}$$

where $G' = (V', E')$, $V' = V \cup \{u_i\}_{i=1}^k$ and $E' = (E \setminus \{(v, x) : x \in S_G(v)\}) \cup \{(v, u_i) : i \in [k]\} \cup \{(u_i, x) : x \in S_G(v), i \in [k]\}$. Here, $\{u_i\}_{i=1}^k$ constitutes the aforementioned committee and $E'$ is obtained from $E$ by deleting the edges between vertex $v$ and its successors, adding edges between vertex $v$ and the committee and adding edges between members of the committee and the original successors of vertex $v$.

**Lemma 1** *PageRank satisfies vote by committee.*

**Proof:** Let $r$ be a stationary distribution for the original graph $G$. That is, $r$ is a solution to $A_G r = r$. We define a new vector $r'$ which, we claim, is the stationary distribution for the new graph up to a positive scalar multiple. Assume WLOG that we replace vertex $v_1$ by a committee of $k$ vertices and assign the committee vertices the indices $n+1, \ldots, n+k$. Let $r' \in \mathbb{R}^{n+k}$ such that

$$r' = (r_1, \ldots, r_n, \frac{r_1}{k}, \ldots, \frac{r_1}{k})^T. \tag{4}$$

The transition matrix for the new graph $G'$ is

$$A_{G'} = \begin{pmatrix} 0 & a_{12} & \ldots & a_{1n} & a_{11} & \ldots & a_{11} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \ldots & a_{nn} & a_{n1} & \ldots & a_{n1} \\ \frac{1}{k} & & & & & & \\ \vdots & & & 0 & & & \\ \frac{1}{k} & & & & & & \end{pmatrix} \tag{5}$$

where $[A_{G'}]_{ij}$ is the probability of going from page $j$ to page $i$. Note that $A_{G'}$ is in conformity with our formal definition of $G'$. It is enough to show that $A_{G'} r' = r'$ since we have $[r'_i \geq r'_j \Leftrightarrow r_i \geq r_j]$ for all $v_i, v_j \in V$ and the relative ordering of the vertices is the same as before. For all $i \in [n]$, we have

$$[A_{G'} r']_i = \sum_{j=2}^{n} a_{ij} r_j + k a_{i1} \frac{r_1}{k} = r_i = r'_i. \tag{6}$$

Furthermore, for all $i \in [n+k] \setminus [n]$, we have

$$[A_{G'} r']_i = \frac{1}{k} r_1 = r'_i. \tag{7}$$

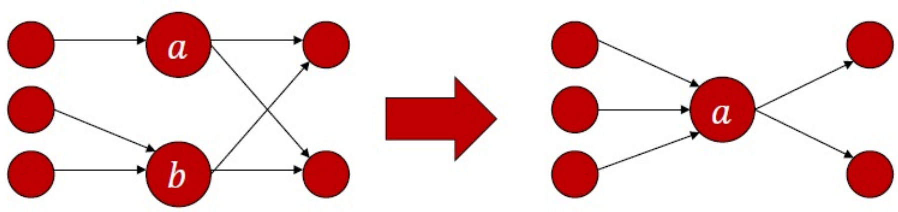
This shows that $r'$ is indeed the stationary distribution of $G'$ and completes the proof. ■

**Axiom 3: Self-Edge**

A ranking system satisfies *self-edge* if adding a self-edge to vertex $v$ strengthens vertex $v$ but does not change the ranking of other vertices. Note that if vertex $v$ was in a tie with another vertex to begin with, adding a self-edge will break the tie in favor of vertex $v$ in this setting.

**Axiom 4: Collapsing**

A ranking system satisfies *collapsing* if vertices that vote identically can be merged into a single vertex, with all the incoming edges of the original vertices, and the ranking of the vertices that were not collapsed remains unchanged.




In the example above, vertices $a$ and $b$ are voting for the same two vertices but they get votes from different vertices. The collapsing axiom says that the ranking of vertices other than vertices $a$ and $b$ stays the same when we go from the configuration on the left-hand side to that on the right by merging vertices $a$ and $b$.

**Axiom 5: Proxy**

A ranking system satisfies *proxy* if $k$ vertices of equal rank that voted for $k$ alternatives via proxy can achieve the same result by voting for one alternative each. Note that, in PageRank, the condition that the $k$ vertices have equal rank is equivalent to the one requiring that they have the same stationary probability.

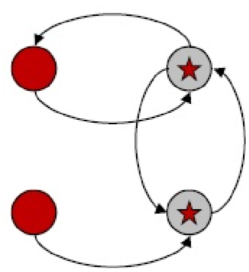
**Theorem 2 (Altman and Tennenholtz, 2005)** *A ranking system satisfies axioms* $1-5$ *if and only if it is the PageRank ranking system.*

Altman and Tennenholtz prove the "only if" direction by showing that the five axioms imply a unique ranking on each graph. They use the axioms (and some nonintuitive corollaries) to transform the input graph into one in which the ordering between every pair of vertices is clear. Because PageRank satisfies the axioms, the unique ranking implied by the axioms must be the one produced by PageRank.

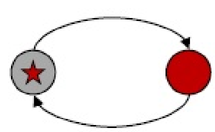
## 3 Selecting a Subset

In this subsection, we again work with directed graphs in which each edge represents a vote of approval/ trust/ support. However, now, instead of ranking the vertices, we want to find a subset of size $k$ which maximizes the sum of the in-degrees of the selected agents. A likely context for this problem is advertising in directed social networks. A $k$-selection system $f$ receives a directed graph as input and outputs $V' \subseteq V$ such that $|V'| = k$. We say that $f$ is *impartial* if $v_i \in f(G)$ independently of the votes of $v_i$. This concept is closely related to strategyproofness. Indeed, suppose a vertex gets a utility of 1 if it is selected and a utility of 0 otherwise. If the $k$-selection system is impartial, no vertex can gain any utility by changing its outgoing edges and the $k$-selection system cannot be manipulated.

Note that a system that always outputs the optimal solution is not necessarily impartial. To see this, consider the example below.



Here, the vertices marked with stars represent an optimal solution to the 2-selection problem with a total in-degree of 4. The vertex at the top left-hand side has only one incoming edge. However, if it removes its only outgoing edge, it has the chance to be in the selected subset under some tie-breaking rule. This suggests the following question: Is there an impartial $k$-selection system which has a (multiplicative) approximation guarantee with respect to the optimal solution? Let us consider how we can answer this question for different values of $k$. For $k = n$, we clearly have no problem. There is a unique solution which will necessarily be output by any $n$-selection system. To gain some insight into what happens in the general case, let us look at the following simple example for $k = 1$:



The optimal solution clearly has total in-degree 1. Furthermore, any 1-selection system will choose one of the two vertices. Suppose we have an impartial 1-selection system which chooses the vertex marked with a star. If the other vertex removes its only outgoing edge, this system must still choose the starred vertex by impartiality. However, now we are in a situation where the optimal solution has value 1 while the best impartial 1-selection system produces a solution with value 0. Thus, we conclude that there cannot be an impartial $k$-selection system with a finite approximation guarantee when $k = 1$. For the general case, we have an impossibility result by Alon et al.

**Theorem 3 (Alon et al. 2011)** *For all $k \in [n-1]$, there is no impartial $k$-selection system with a finite approximation ratio.*

**Proof:** We prove the theorem for the case $k = n - 1$ only. Suppose there exists an impartial $k$-selection system with a finite approximation ratio. It is enough to show that this rule cannot have the property that if there is a unique vertex with incoming edges, it is not eliminated. When the input is an empty graph on $n$ vertices, we can assume WLOG that vertex $v_n$ is eliminated. Now, consider graphs on $n$ vertices which consist of a star with vertex $v_n$ at the center and other isolated vertices. Vertex $v_n$ can not be eliminated in this case because the $k$-selection system has a finite approximation ratio. Let $f : (\{0,1\}^{n-1} \setminus \{0\}) \to [n-1]$ be a function which maps a vector of the out-degrees of the vertices $\{v_i\}_{i=1}^{n-1}$ to the index of the vertex to be eliminated. By impartiality, we must have $f(x) = i \Leftrightarrow f(x + e_i) = i$ for all $i \in [n-1]$ where the addition $x + e_i$ is done modulo 2. Because the elements of the set $f^{-1}(i)$ come in pairs, $|f^{-1}(i)|$ must be even for all $i \in [n-1]$. This implies that $|\text{dom}(f)|$ must be even as well. However, we have $|\text{dom}(f)| = 2^{n-1} - 1$, a contradiction. ■

This motivates an inquiry into what we can achieve via randomization. Let us consider a simple system in which we assign the vertices of the graph to $m$ subsets uniformly and independently at random. In each subset, we then select roughly $\frac{k}{m}$ vertices with the highest in-degrees only based on edges coming from other subsets. We call this a randomized $m$-partition system. Note that the output of this system is a probability distribution over impartial systems because the votes cast by a vertex only affect which vertices are selected in *other* subsets. Furthermore, we can prove that it enjoys some approximation guarantees.

**Theorem 4 (Alon et al. 2011)**

1. *The approximation ratio is $4$ with $m = 2$.*
2. *The approximation ratio is $1 + O\left(\frac{1}{k^3}\right)$ for $m \sim k^{\frac{1}{3}}$.*

**Proof:** We prove only (1). Assume for ease of exposition that $k$ is even. Let $K$ be the optimal set. A partition $\pi = (\pi_1, \pi_2)$ divides $K$ into two subsets $K_1^\pi = K \cap \pi_1$ and $K_2^\pi = K \cap \pi_2$. Let $d_1^\pi = |\{(u,v) \in E : u \in \pi_2, v \in K_1^\pi\}|$ and $d_2^\pi = |\{(u,v) \in E : u \in \pi_1, v \in K_2^\pi\}|$. Note $|K_1^\pi| \leq k$. Because we select the $\frac{k}{2}$ vertices with the highest in-degrees based on the edges coming from other subsets, we capture at least $\frac{d_1^\pi}{2}$ edges coming into $\pi_1$. Similarly, we capture at least $\frac{d_2^\pi}{2}$ edges coming into $\pi_2$. Furthermore, we have

$$\mathbb{E}(d_1^\pi + d_1^\pi) = \frac{\text{OPT}}{2} \tag{8}$$

since the random partition selects each edge to be in the cut $[\pi_1, \pi_2]$ with probability $\frac{1}{2}$. Hence,

$$\mathbb{E}(\text{ALG}) \geq \mathbb{E}(\frac{d_1^\pi + d_1^\pi}{2}) = \frac{\text{OPT}}{4}. \tag{9}$$

■

Note that increasing the number of subsets brings an improvement in the approximation guarantee as it allows the system to take more edges into account when it selects vertices.