Algorithms, Games, and Networks January 15, 2013

Lecture 1

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1 Overview and Procedural Stuff

This course is going to be about the theory and algorithms of systems with interacting agents, each with their own intests in mind. For example, we'll talk about auctions, routing, fair division, and more general games.

- Basics of game theory, equilibria
- Quality of equilibria: price of Anarchy
- Social choice: voting, manipulation
- Mechanism design: designing rules of the game to acheif desired outcome, auctions
- Kidney echange, matching markets
- Social networks
- Fair division

The course book will be "Algorithmic Game Theory", which is available freely on the web. The required work for the course is:

- 4 homeworks
- A final project
- Scribing one lecture * Find the link on the course webpage.
- Helping grade one HW
- Participation in class

There's also a Piazza for the course, which you can also find on the course webpage.

2 A Basic Introduction to Game Theory

The field of game theory was developed by economists to study social and economic interactions. For example, they wanted to study the rationale for people's actions in economic situations. What are the effects of incentives? A *game* is an interaction between parties with their own interest. Computer scientists care about game theory because many of today's systems or platforms (such as the internet, or e-mail) involve self-interested agents interacting with each other, which can cause all sorts of weird scenarios, good and bad.

The basic setting in which we'll be working will have *players*, or participants. There will be a set of *strategies*, or choices that each player may choose among. Each player gets some *payment* or payoff, which is a function of the combined strategy choices of all the players.

Our first example is the "Walking on the Sidewalk Game." Suppose there are two players walking towards each other. They can each choose to walk on either their left or their right. If each chooses to walk on their left (or right), then they can pass by each other without difficulty. If, however, one player chooses their left and the other player chooses their right, they either run into each other or have to shift back and forth, an undesireable outcome for both players. Here is a way we can encode this sort of game, listing one player's options as rows and one player's options as columns:

Payoff Matrix

Column Player

		Left	Right
Row Player	Left	1 1	-1 -1
Row Player	Right	-1 -1	1 1

The leftmost number in each cell represents the payment for the row player, and the rightmost number represents the payment for the column player. The tuple of row and column actions determines which cell the payments come from.

A key notion we'll discuss in this class of that of a *Nash Equilibrium.* Formally, a *Nash equilibrium* is a tuple of actions (or a tuple of *distributions* over actions), one entry for each player, such that any player, given everyone else plays their part of the tuple, has no incentive to deviate from their own part of the tuple.

There are three Nash equilibria in this game: everyone walking on the right, everyone walking on the left, and each person choosing left and right with probability 1/2. The last Nash equilibrium is called a *mixed Nash*, while the first two are called *pure Nash*, because the first two use *pure* or deterministic strategies, while the other Nash has players playing *mixed* strategies, or randomizing over strategies.

Another classic example of a game is called the Prisoner's Dilemma. Consider there are two companies that are trying to decide whether they should install polution controls. If it costs \$4 to install polution control for the player who installs it, and each player benefits from breathing cleaner air (and paying less in worker's comp) by \$3 for each polution control installed. So, here is the payoff matrix for this game:

		Firm B	
		Install	Don't Install
Firm A	Install	2 2	3 -1
	Don't Install	-1 3	0 0

As it turns out, there is only one Nash equilibrium in this game: everyone deciding not to install the polution controls. If both firms have polution controls, either of the firms would make more money removing their controls. If just one firm has polution controls, that one firm would stop losing money if they removed their polution controls. If neither firm installs polution controls, neither firm has incentive to install them. This game is also known as the Tragedy of the Commons. Also, notice that, regardless of what the other firm is doing, a firm does best if they choose not to install polution controls. This is an example of an *dominant strategy*: it is the best action to take regardless of what the other player chooses to do.

Another game is one called Matching Pennies/Penalty Shot. Simultaneaously, the goalie and the shooter have to decide whether to go left or right. If they both go left or right, no goal is made; if one goes left and the other goes right, a goal is made and the goalie loses a point for his team and the shooter earns a score for his team. Here is the payoff matrix for this game:

		Shooter	
		Left	Right
Goalie	Left	0 0	1 -1
	Right	1 -1	0 0

This game is somewhat different than the previous games we've talked about. If either player is deterministically playing left or Right, the other player has incentive to be deterministic (in the case of the goalie, in the same direction as the shooter, in the case of the shooter, to the opposite direction of the goalie). This means there isn't a deterministic pair of strategies where one player won't have incentive to deviate. In fact, the only Nash equilibrium in this game is the randomized pair of strategies where each player goes left or right with probability 1/2. To see that no other mixed Nash exists, notice that if either player is slightly biased, then the other player has incentive to play deterministically one way or the other.

These equilibra were named after John Nash. In 1950, he proved the following theorem.

Theorem 1 *Every game admits a mixed Nash equilibrium, if there is a finite number of players and a finite number of strategies for each player to choose between.*

Let's introduce some notation because this is already cumbersome to write out in words. In general, there will be a payment matrix for each of the players (we've been superimposing them in the previous examples). Suppose we have just two players, the row player and the column player. Say the payoff matrix for the row player is called R, and the the payoff matrix for the column player is called C. Then, (p, q) (where p is a probability distribution over actions for the row player and similarly for q and the column player) is a Nash equilibrium if

$$p^T Rq \geq e_i^T Rq \quad \forall i$$

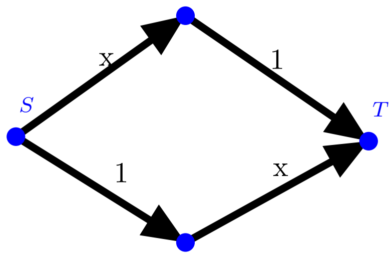
and

$$p^T Cq \geq p^T Qe_j \quad \forall j$$

In words, no particular deterministic action for either the row or column player increases the (expected) payoff for the row or column. This implies there is no benefit for either the row or column player to shift their probability distributions from p or q, respectively. This also implies that any i, j with $p_i, p_j > 0$ must have equal expected payoff, or there would be incentive to move towards the one has the larger expected payoff. Similarly, this is true for all $q_i, q_j > 0$.

There are some strange things which can occur with Nash equilibria. For example, Braess paradox has a bunch of players who are playing on a road network. Everyone is trying to get from s to t, and players choose between paths in the network. Some roads have infinite capacity, so their is now slow-down if lots of people use the road, while other roads will get traffic build-up (where the time to travel along the road is a function of the number of players who choose to take the road).

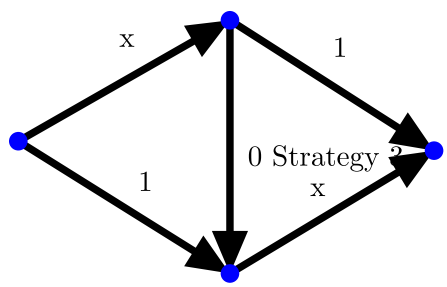
Strategy 2



Strategy 1

Think of x as the fraction of players using that edge. The Nash equilbrium in this case is half of the players taking each path. The paradox comes in if one adds a superhighway between the middle two nodes, with infinite capacity (so there is zero cost for taking the superhighway):

Strategy 2



Strategy 1

In this case, in a Nash equilibrium you will have to have all players (or all but one) taking both of the x edges. This means that even though we increased capacity, the travel time for every player has *increased* from 1.5 to 2. It is a bit like the prisoner's dilemma.

2.1 Two-player zero-sum games

We'll now talk about games with 2 players, where the sum of the payoffs for the row and column players, for any pair of strategies, is 0. These are called *zero-sum games*. These games need *not* be fair; for example, the penalty shot game isn't "fair" since you'd rather be taking the shot than receiving it. Zero-sum just means that any time one person gains some amount, the other player loses that amount.

A *minimax-optimal strategy* is a randomized strategy that has the best guarantee on its expected gain, over all possible choices of the opponent. That is, this is the strategy you should play if your opponent knows you well: if he is to best-respond to whatever strategy you choose, your strategy is maximizing your expected payoff (and, similarly, minimizing her expected payoff, since this is zero-sum). In the case of the shooter game, the 50/50 strategy for each of the players is *minimax-optimal.*

If we have a goalie who's weaker on his left, the payoff might look like this:

Payoff Matrix

Goalie

		L	R
Shooter	L	-1/2 1/2	-1 1
	R	-1 1	0 0

What is the minimax-optimal pair of strategies for this modified game? Place probability p on the shooter going left, and $1-p$ on the shooter going right. Then, one can solve and see that the shooter gets the best worst-case guarantee for $p = 2/3$. In this case, the ball has a 2/3 probability of going in whichever way the goalie dives. It's also the case that the goalie has a minimax optimal strategy to go left 2/3 of the time. This guarantees that the ball has at *most* a 2/3 probability of going in, whichever way the shooter shoots.

In general, one can find the minimax-optimal (pair of) strategies in a larger game using linear programming. We'll have variables $p_1, \ldots, p_k, v$, and use the following LP:

$$
\begin{aligned}
&\text{maximize } v \\
&\text{subject to} \\
&p \cdot M_j \geq v \ \ \forall j \\
&\sum_{i=1}^{k} p_i = 1 \\
&\quad p_i \geq 0 \ \ \forall i
\end{aligned}
$$

The game we analyzed above had the property that there was a value v (in that case, 2/3) such that the row player had a (randomized) strategy guaranteeing an expected gain at least

v no matter what column the column-player chose, and similarly the column-player had a (randomized) strategy guaranteeing an expected loss at most v no matter what row the row-player chose. In other words, if you're optimal and playing against an optimal opponent, it doesn't hurt to reveal your (randomized) strategy. This in fact holds in general, and is the *minimax theorem* proved by Von Neumann in 1928.

Theorem 2 *Every 2-player, zero-sum game has a unique value v, and minimax optimal strategies p and q for each player such that the row player can guarantee payoff at least v playing p and the column player can guarantee her loss to be at most v playing q.*

We can actually prove the minimax theorem using the existence of Nash equilibria in 2-player, zero-sum games.

Proof: Pick some NE (p, q) and let V be the value to the row player in that equilibrium (and $-V$ is what the column player gets). Neither player can do better by deviating from their strategy, even knowing the strategy of the other player (since it's a Nash Equilibrium). Since the column player can't do any better by deviating, this means (because the game is zero-sum) that by playing p, the row player is guaranteed an expected gain $\geq V$ no matter what the column player does. Similarly, since the row-player can't do any better by deviating, this means that by playing q, the column player is guaranteed an expected loss $\leq V$ no matter what the row-player does. So, they are both playing minimax optimal strategies. ■

There are much more constructive arguments for the minimax theorem, which we'll look at soon. We can, however, solve for minimax optimal strategies in polynomial time. On the other hand, solving for Nash equilibria is PPAD-hard, and a slough of other questions relating to the properties of Nash equilibria of a certain game are NP-hard.

Can one use the notion of minimax optimality to understand some game like poker? Consider Kuhn poker. There are three cards $1, 2, 3$, and two players A, B. Each players antes \$1. Each player gets one card. A goes first. Can bet \$1 or pass. If A bets, B can call or fold. If A passes, B can bet \$1 or pass. If B bets, A can call or fold. Then, whoever has the highest card wins (unless a player folds, in which case the other player wins).

If we wanted to write this game as a matrix, we need to think about the interactions like a program. For a given card, A can decide to do one of the following:

- Pass but fold if B bets
- Pass and bet of B bets
- Fold

So, A will have 9 possible actions: for each card, can choose one of the three above actions. A could randomize over these strategies, too. Similarly, B can write a program-like strategy out as a function of his card and what A does.

The minimax-optimal strategies for this are:

For A:

- If hold 1, then $\frac{5}{6}$ PassFold and $\frac{1}{6}$ Bet
- If hold 2, then $\frac{1}{2}$ PassFold and $\frac{1}{2}$ PassCall
- If hold 3, then $\frac{1}{2}$ and $\frac{1}{2}$ Bet

For B:

- If hold 1, then $\frac{2}{3}$ FoldPass and $\frac{1}{3}$ FoldBet
- If hold 2, then $\frac{2}{3}$ FoldPass and $\frac{13}{}$ CallPass.
- If hold 3, then CallBet

This game has slightly negative value for A: her expected payoff is $-1/18$.

2.2 Proof of existence of Nash equilibria

How could we prove the existence of the Nash Equilibria? We'll use a big hammer called Brouwer's Fixed-point theorem.

Theorem 3 *Let S be a convex compact region in $\mathbb{R}^n$ and let $f : S \to S$ be a continuous function. Then there must exist $x \in S$ such that $f(x) = x$. x is called a fixed-point of f.*

On a line segment, or 1-space, this theorem is intuitive. This theorem is highly non-obvious in n-space. Notice that while it's true for $[0, 1]$, it is not true for $(0, 1)$ — e.g., consider $f(x) = x/2$. It's also not true for the infinite line — e.g., consider $f(x) = x + 1$. Also while it is true for the circle with interior included $\{(x, y) : x^2 + y^2 \leq 1\}$, it is not true for the circle without the interior $\{(x, y) : x^2 + y^2 = 1\}$ — e.g, consider $f((x, y)) = -(x, y)$.

Now, we'll prove the existence of Nash equilibria for two-player games of finite action-space size.

Proof:

We will talk about $S = \{(p, q) : p, q \text{ are legal probability distributions on } 1, \ldots, n\}$. Ie, S is the set of pairs of mixed strategies. This is a convex compact set. We want to define some $f(p, q) = (p', q')$ such that f is continuous, and any fixed point of f is a Nash equilibrium. Then, we can apply Brouwer's fixed-point theorem and have a proof.

Our first attempt is $f(p, q) = (p', q')$ where p' is the best response to q and q' is the best-response to p. This has two problems. First, the function isn't continuous (think about the goalie game; a very slight change in one player's strategy can make the best-response for another to shift their weight entirely to L or R) and is also not well-defined. Instead, let's try $f(p, q) = (p', q')$ such that

- q' maximizes [(expected gain w.r.t p)$- \| q - q' \|^2$]
- p' maximizes [(expected gain w.r.t q)$- \| p - p' \|^2$]

You can think of this as penalizing p' and q' for moving too far from p and q. This is well-defined, because both players are maximizing a negative quadratic. Moreover, it's continuous because quadratic and linear functions are continuous.

So, it remains to show that a fixed-point of f is a Nash equilibrium. This is easiest to see by contrapositive. Suppose (p, q) is not a Nash equilibrium. This means that for at least one of the players (say, the column player), there is some point on the simplex she would rather be than her current strategy. This means the linear term above has positive slope in some direction along the simplex. Since the second (quadratic) term has derivative zero at zero, this means the maximum of the sum (linear + quadratic) must occur at a point of nonzero distance. So, we are not at a fixed point. ■