Proofs for Lecture 13

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Theorem 1. *Let there be a CSP with $|D| = d$ and arity r (each constraint having at most r variables). If it is strong $(d(r-1)+1)$-consistent, then it is globally consistent.*

Proof. For simplicity we provide the proof for the special case of $r = 2$.

We will prove the theorem by showing that strong $(d+1)$-consistent binary CSPs are $(d+i+1)$-consistent for any $i \geq 1$.

According to the definitions, we need to show that if $\bar{x} = (x_1, ...x_{d+i})$ is any locally consistent subtuple of the subset of variables $\{X_1, ..., X_{d+i}\}$, and if X_{d+i+1} is any additional variable, then there is an assignment x_{d+i+1} to X_{d+i+1} that is consistent with $\bar{x}$.

We call an assignment to a single variable a unary assignment and we view $\bar{x}$ as a set of such unary assignments. With each value $j \in D$ we associate a subset A_j that contains all unary assignments in $\bar{x}$ that are consistent with the assignment $X_{d+i+1} = j$. Since variable X_{d+i+1} may take on d possible values $1, 2, ...d$ this results in d such subsets, $A_1, ..., A_d$.

We claim that there must be at least one set, say A_1, that contains the set $\bar{x}$. If this were not the case, each subset A_j would be missing some member, say x'_j, which means that the tuple generated by taking a missing unary assignment from each of the A_j's , i.e. $\bar{x}' = (x'_1, x'_2, ..., x'_d)$ whose length is d or less (there might be repetitions), could not possibly be consistent with any of X_{d+i+1}'s values.

This leads to a contradiction because as a subset of $\bar{x}$, $\bar{x}'$ is locally consistent, and from the assumption of strong $(d+1)$-consistency, this tuple should be extensible by any additional variable including X_{d+i+1}.

Note that we need not assume that the x'_i's are distinct unary assignments because strong $(d+1)$-consistency renders the argument applicable to subtuples $\bar{x}'$ of length less than d.

We found a subset, without loss of generality A_1, that contains the set $\bar{x}$. From the definition of A_1, it is consistent with $X_{d+i+1} = 1$. Hence, we found a value consistent with $\bar{x}$. □

Theorem 2. *Let there be a CSP with arity r. Let t be an upper bound on the number of constraints each variable appears in. Let q be a lower bound on the probability of choosing a satisfying assignment for a constraint. If $q \geq 1 - \frac{1}{e(r(t-1)+1)}$ then there is a solution to the CSP.*

Lemma 3 (Lovász Local Lemma)**.** *We denote by $\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n$ the series of events such that each event occurs with probability at most p and such that each event is independent of all the other events except for at most m of them. If $ep(m+1) \leq 1$ (where $e = 2.718...$), then there is a nonzero probability that none of the events occur, $\Pr[\bigcap_{i=1}^{n} \bar{\mathcal{E}}_i] > 0$.*

*Based on lecture notes by Zvi Vlodavsky and Bracha Hod.

Proof of Theorem 2. Let there be a random assignment of variables. $\mathcal{E}_i$ is the event of C_i not being satisfied. Since a constraint has at least a q probability of being satisfied, $\Pr[\mathcal{E}_i] \leq 1 - q$. Since a constraint has at most r variables, each appearing in at most $(t-1)$ other constraints, $\mathcal{E}_i$ is independent of all other events except for at most $r(t-1)$ events.
According to The Lovász Local Lemma, by assigning $p = 1-q$, $m = r(t-1)$, if $e(1-q)(r(t-1)+1) \leq 1$ then $\Pr[\bigcap_{i=1}^{n} \bar{\mathcal{E}}_i] > 0$. Hence, there is a solution to the CSP. □