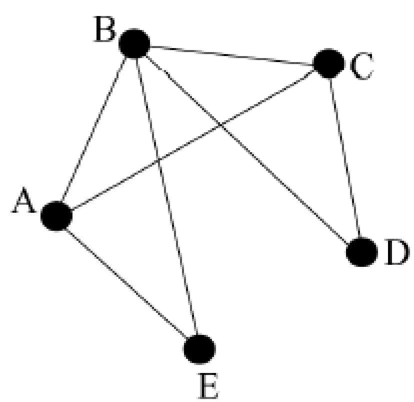
Reasoning with uncertainty

- (Very) basic review of probability and uncertainty
- Joint distribution and inference
- Exploiting independences
- Special case: Directed graphs
- General case: Undirected graphs, factor graphs
- Examples

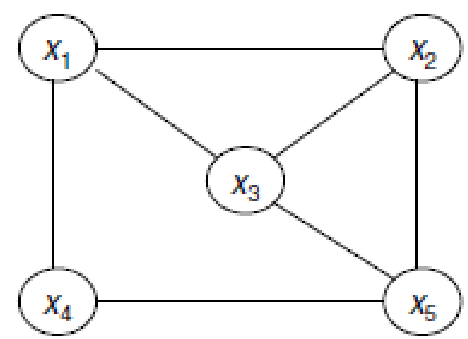
- Deterministic:
  - Represent facts and constraints
  - Find configuration that satisfies representation
- Examples:
  - Clauses $A \wedge B \Rightarrow C$
  - Satisfiability problems

$\varphi = \{(\neg C), (A \vee B \vee C), (\neg A \vee B \vee E), (\neg B \vee C \vee D)\}.$



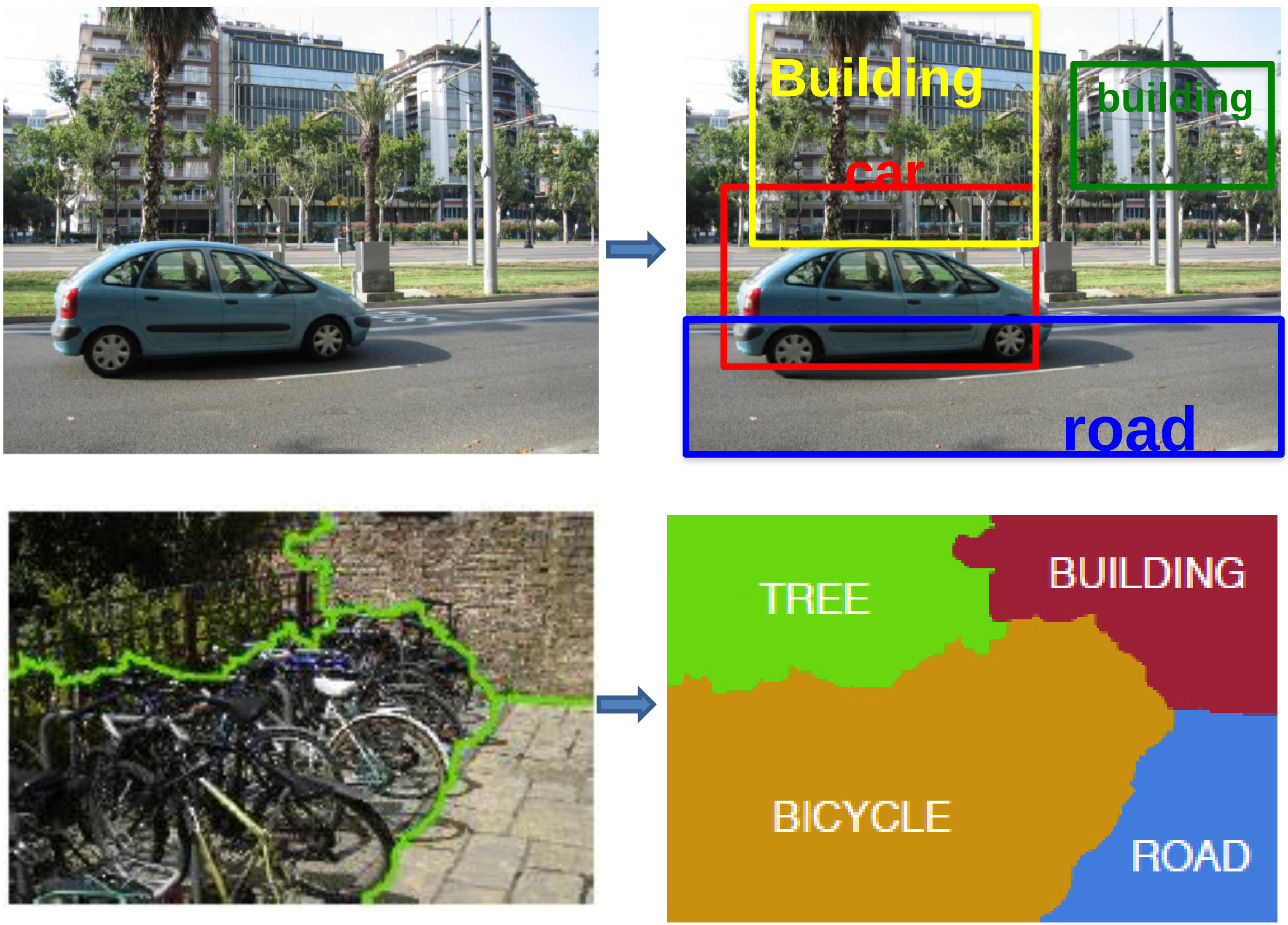
$f_1(x_1, x_2, x_3)$
$f_2(x_2, x_3, x_5)$
$f_3(x_1, x_4)$
$f_4(x_4, x_5)$

$$\min_{t \in Sol}\{\sum_{i=1}^{m'} f_i(t)\}$$



- Generalization to include uncertainty due to imperfect knowledge
  - Variables: Deterministic → Random variables
  - Constraints: Deterministic functions (e.g., CNF, CSP, SAT) → continuous output
- Similar problems, generalization

- What we have: scores from noisy classifiers from local features (P(label|image features))
- What would like:



- What we get:

Reasoning

- Need to use knowledge about the world
- Need to integrate uncertainty in “sensing”
- Scenes:
  - “road scenes” contain “cars”, “building”….
  - “Office scenes” contain “desks”, “computers”…
- Co-occurrence:
  - “keyboard” implies “mouse”
- Location:
  - “Cars” are on top of “roads”
  - “Sky” is above “buildings”

Reasoning with uncertainty

- Need use *uncertain* knowledge about the world
- Scenes:
  - “road scenes” *is likely to* contain “cars”, “building”….
  - “Office scenes” *is likely to* contain “desks”, “computers”…
- Co-occurrence:
  - “keyboard” *usually* implies “mouse”
- ……
- Probably possible to represent uncertainty on each individual piece of knowledge
- Intractable to integrate them all to find the “optimal” interpretation

Probability Reminder

- Conditional probability for 2 events A and B:

$$P(A|B) = \frac{P(A,B)}{P(B)}$$

- Chain rule:

$$P(A,B) = P(A|B)\,P(B)$$

Probability Reminder

- Conditional probability for 2 variables X and Y:

$$P(X=x \mid Y=y) = \frac{P(X=x,Y=y)}{P(Y=y)}$$

- Chain rule:

$$P(X=x,Y=y) = P(X=x|Y=y)\ P(Y=y)$$

- For any values x,y

The Joint Distribution

- Joint distribution = collection of all the probabilities P(X = x,Y = y,Z = z…) for all possible combinations of values.
- For m binary variables, size is 2^m
- Any query can be computed from the joint distribution

X	Y	Z	Prob
T	T	T	0.1
T	T	F	0.22
T	F	T	0.2
T	F	F	0.08
F	T	T	0.1
F	T	F	0.15
F	F	T	0.07
F	F	F	0.08

The Joint Distribution

- Any query can be computed from the joint distribution
- Marginal distribution

  $P(X = True), P(X = False)$

- Conditional distribution:

  $P(X = True \mid Y = True) =$

  $P(X = True, Y = True)/P(Y = True)$

- In general:

  $P(E_1 \mid E_2) = P(E_1, E_2)/P(E_2)$

  $$P(E_2) = \sum_{\text{Entries that match } E_2} P(\text{Joint Entries})$$

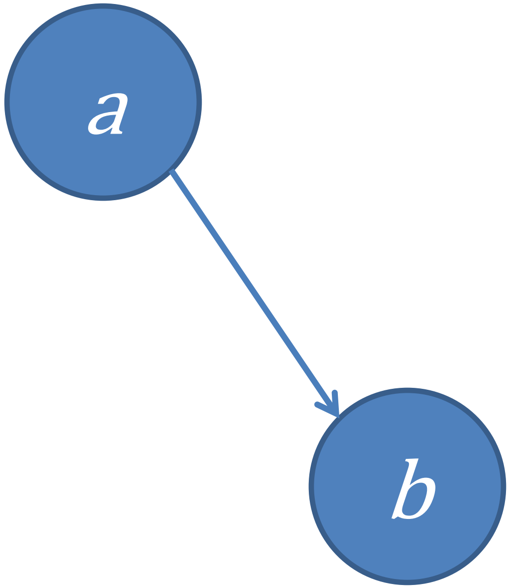
X	Y	Z	Prob
T	T	T	0.1
T	T	F	0.22
T	F	T	0.2
T	F	F	0.08
F	T	T	0.1
F	T	F	0.15
F	F	T	0.07
F	F	F	0.08

Summary

- Any query computable from
- Sum rule:
- Product rule

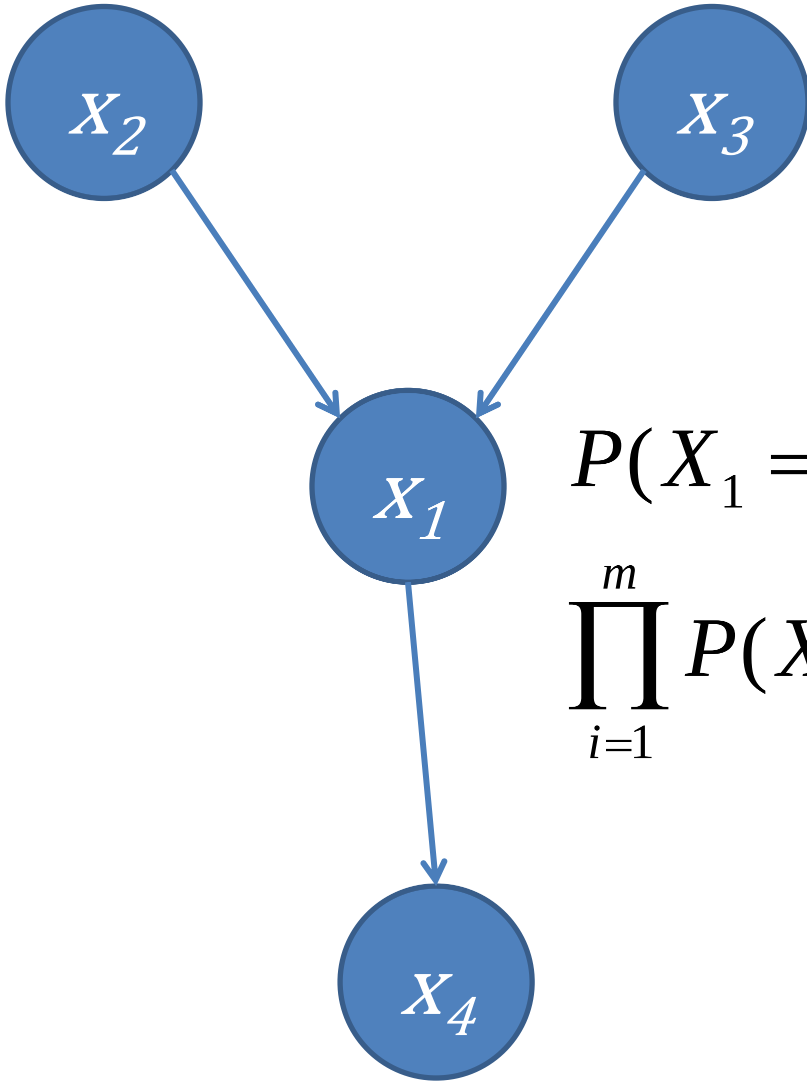
But requires entire joint distribution →
Represent dependencies between variables

First case: Directed



$$P(a, b) = P(b|a)P(a)$$

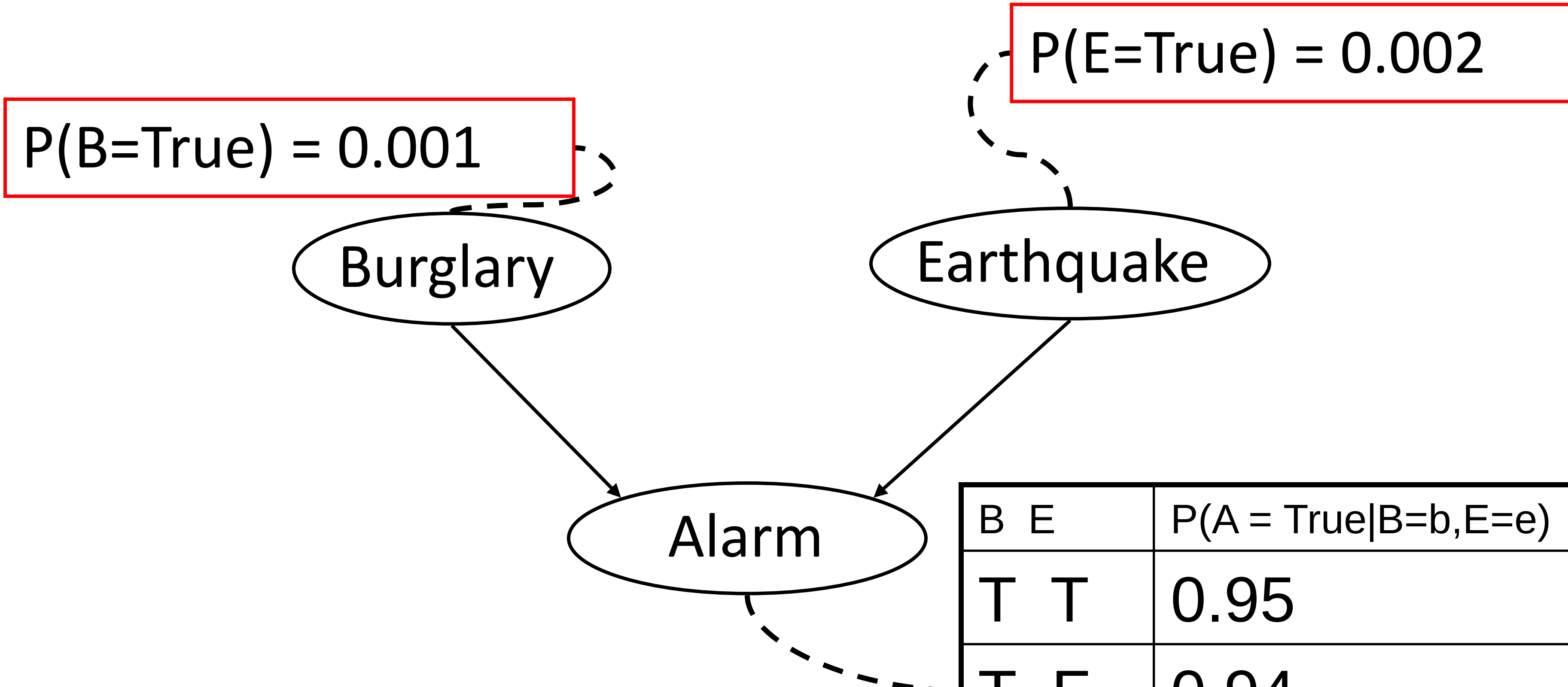
First case: Directed



$$P(X_1 = x_1, X_2 = x_2, \ldots X_m = x_m) =$$

$$\prod_{i=1}^{m} P(X_i = x_i \mid \mathrm{Parents}(X_i))$$

Graphical Representation



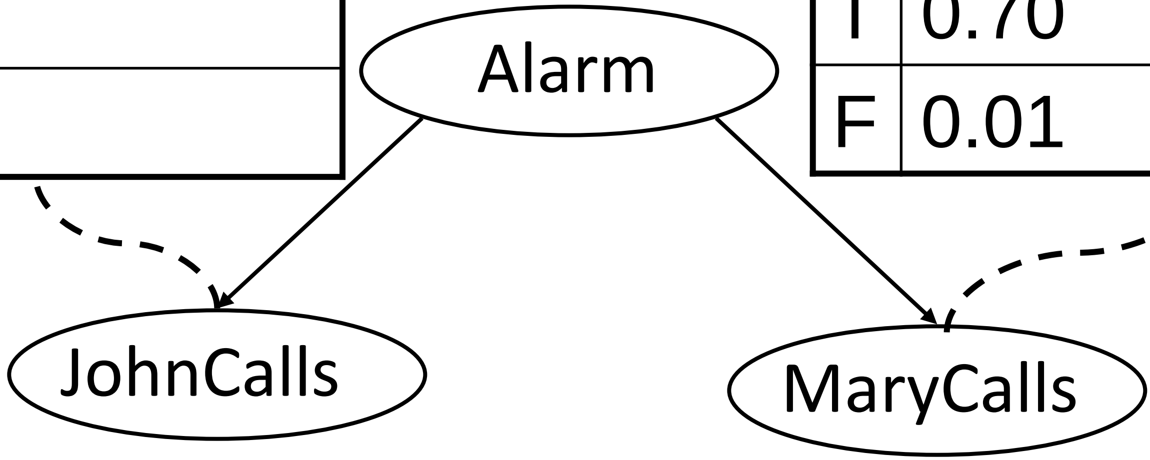
B	E	P(A = True|B=b,E=e)
T	T	0.95
T	F	0.94
F	T	0.29
F	F	0.001

$$P(A,J,M) = P(A|B,E)P(B)P(E)$$

Graphical Representation

A	P(J = True|A=a)
T	0.90
F	0.05

A	P(M = True|A= a)
T	0.70
F	0.01



Given knowledge of A, knowing anything else in the diagram won't help with J and M

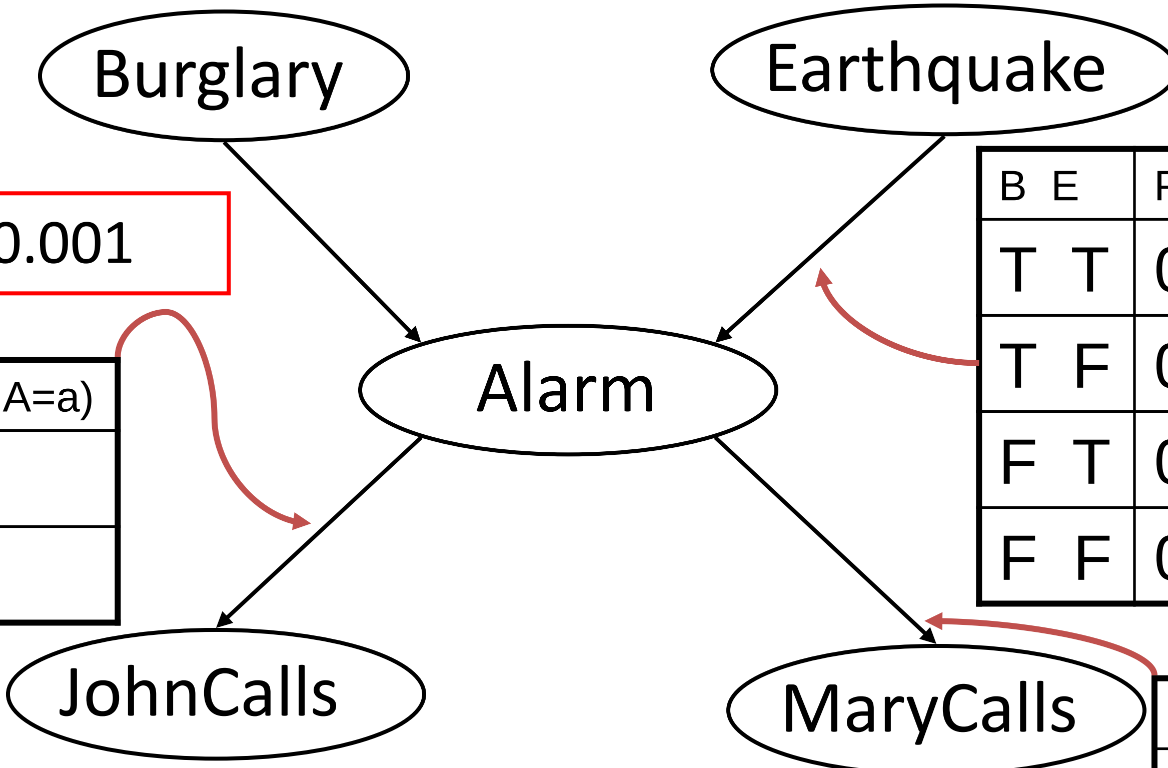
$$P(A,J,M) = P(A)P(J|A)P(M|A)$$

Inference

- *Any* inference operation of the form P(values of some variables | values of the other variables) can be computed

P(E=true) = 0.002

P(B=True) = 0.001



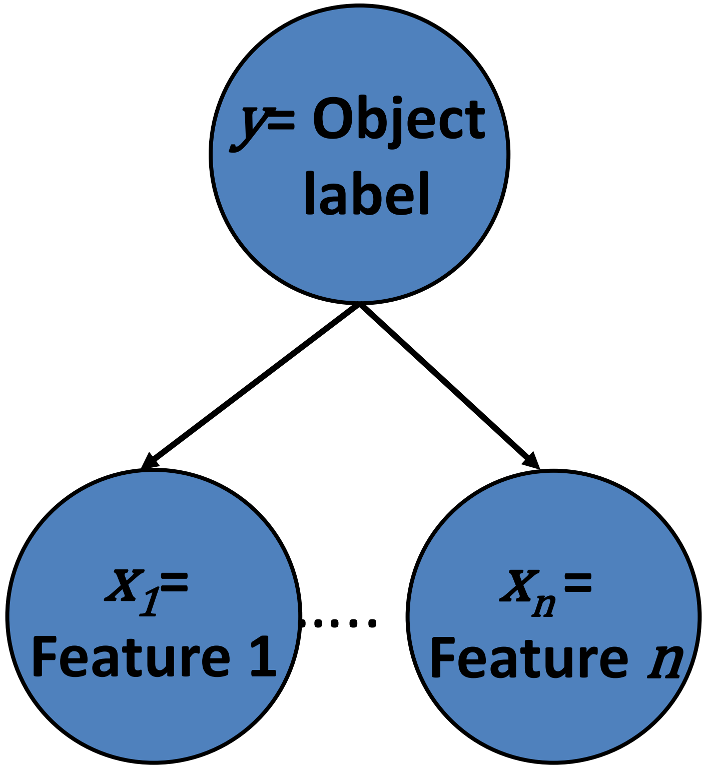
B	E	P(A = True|B=b,E=e)
T	T	0.95
T	F	0.94
F	T	0.29
F	F	0.001

A	P(J = True|A=a)
T	0.90
F	0.05

A	P(M = True|A=a)
T	0.70
F	0.01

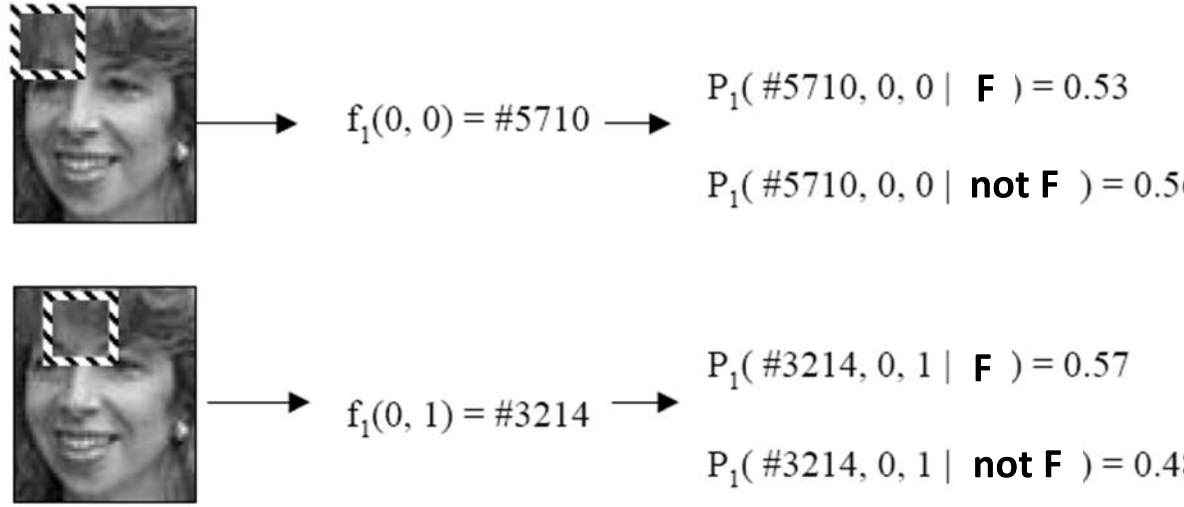
$$P(A,J,M,B,E) = P(B)P(E)P(A|B,E)P(J|A)P(M|A)$$

Example Naïve Bayes classification



$$P(x_1,..,x_n|y) = \prod_i P(x_i|y)$$

Example



$y = 1$ if face

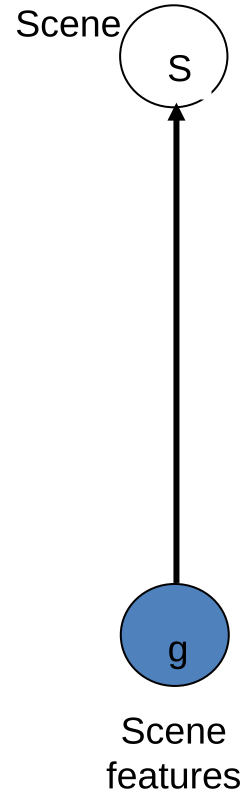
- Lots of (discretized) features from local filters
- Estimate likelihood ratio

$$\frac{P(x_1,..,x_n|y=face)}{P(x_1,..,x_n|y=not\ face)}$$

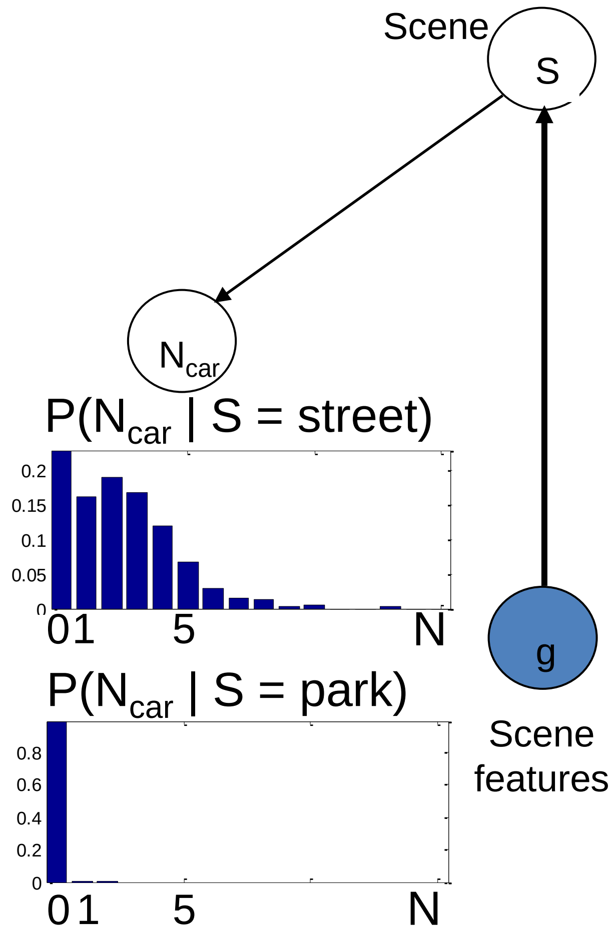
Example

- Need use *uncertain* knowledge about the world
- Scenes:
  - "road scenes" *is likely to* contain "cars", "building"….
  - "Office scenes" *is likely to* contain "desks", "computers"…
- Co-occurrence:
  - "keyboard" *usually* implies "mouse"
- ……
- Probably possible to represent uncertainty on each individual piece of knowledge
- Intractable to integrate them all to find the "optimal" interpretation

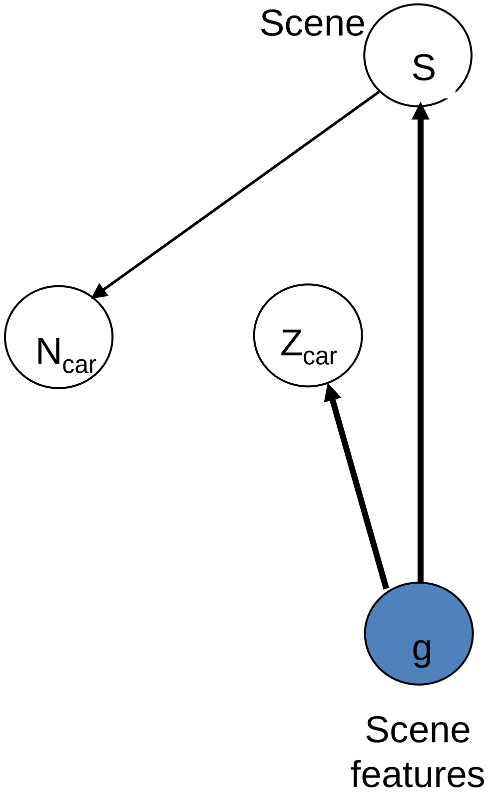
Examples in the next few slides from: Murphy, Torralba, Freeman; NIPS 2003. Torralba, Murphy, Freeman, CACM 2010.



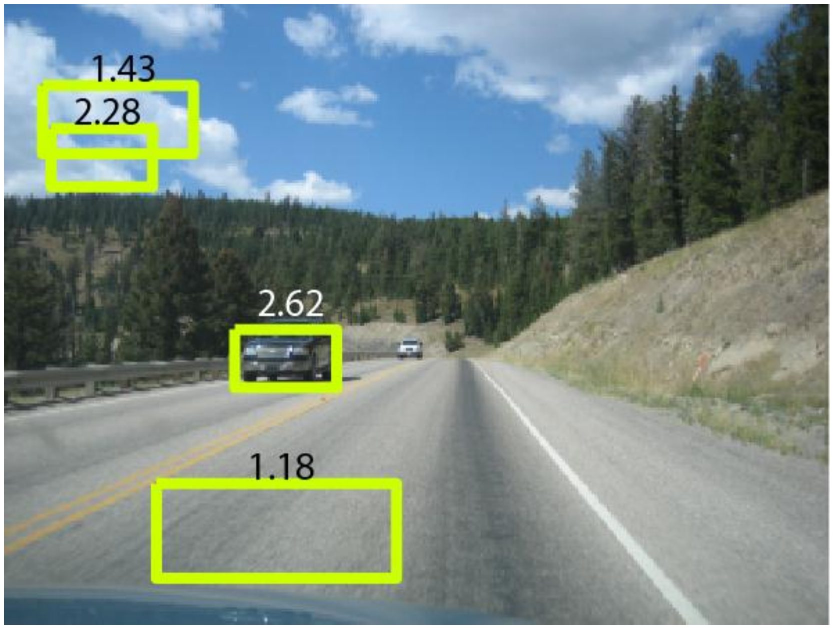
Murphy, Torralba, Freeman; NIPS 2003. Torralba, Murphy, Freeman, CACM 2010.



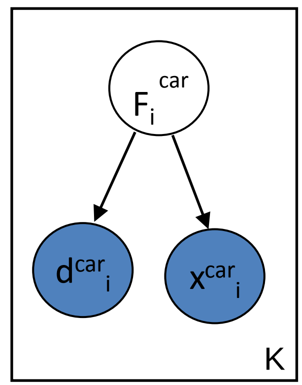
Murphy, Torralba, Freeman; NIPS 2003. Torralba, Murphy, Freeman, CACM 2010.


Scene
S
N_{car}
Z_{car}
g
Scene
features


STOP



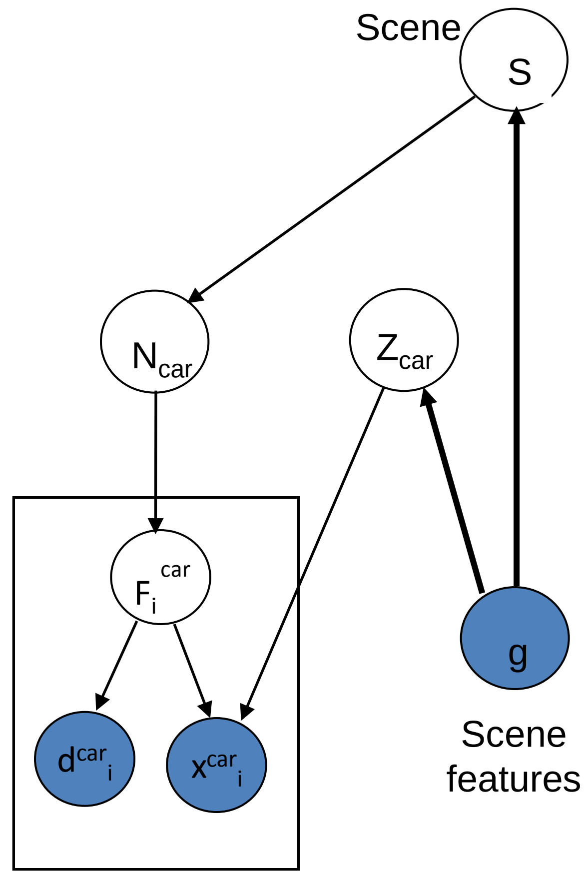
Multiview car detector.

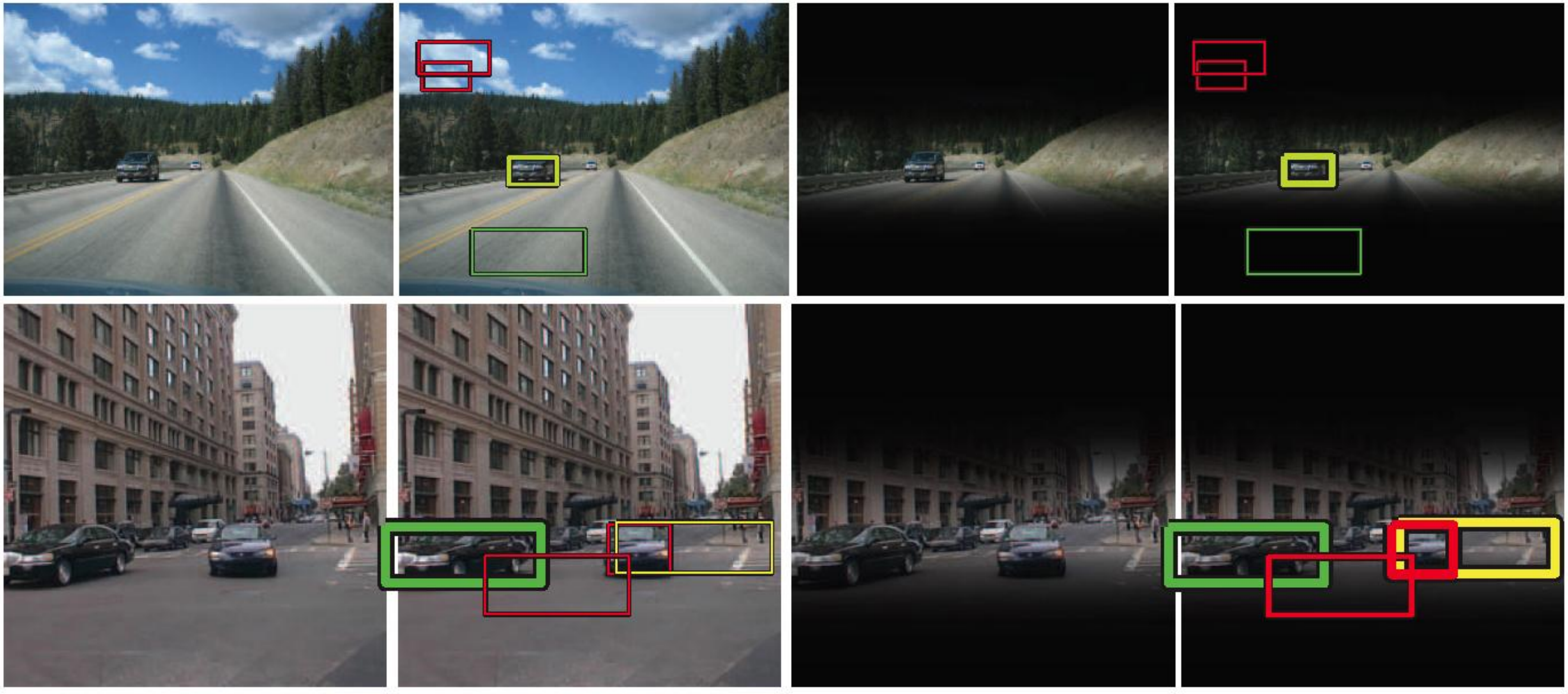


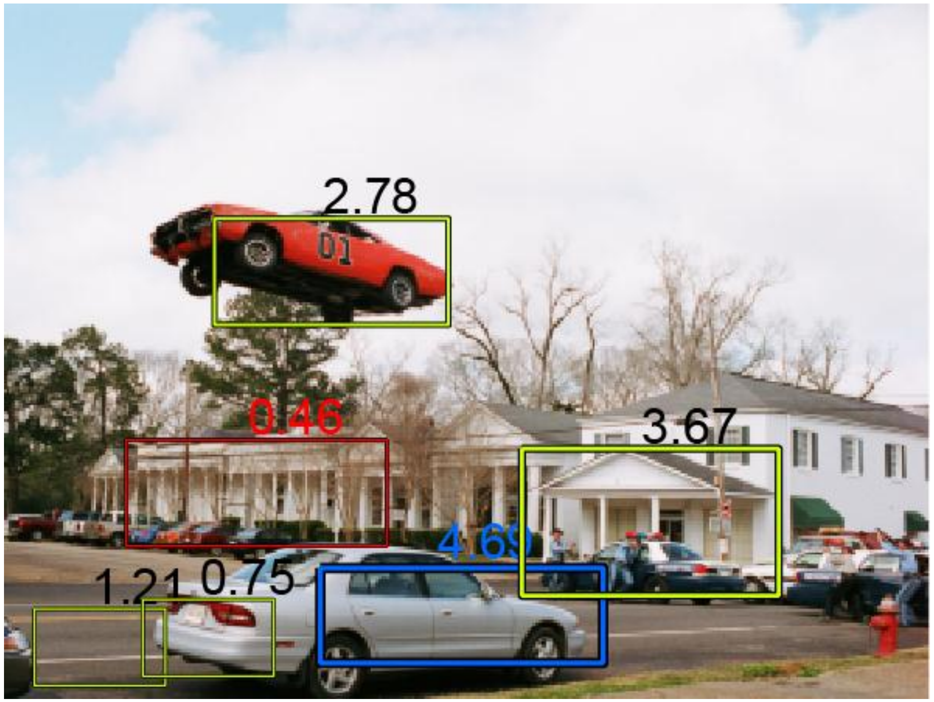
F = 1 if car present in box

p(d | F=1)

An integrated model of Scenes, Objects, and Parts







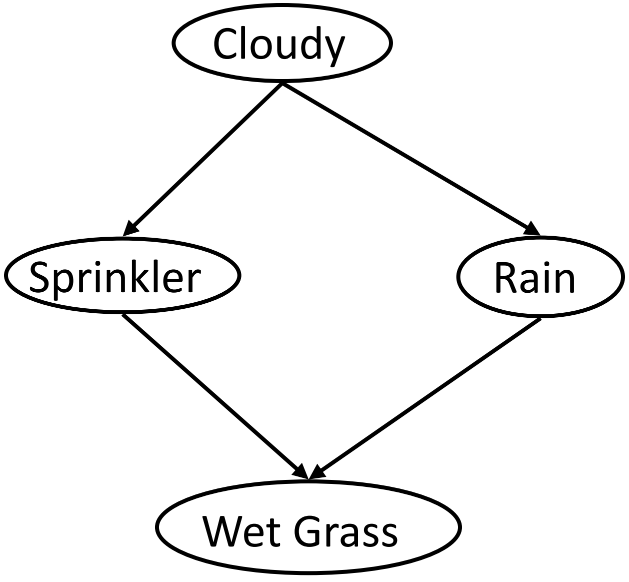
- No miracle: Fancy representation can only model the knowledge that we encoded.

Example from Antonio Torralba

Inference

- Can answer any query but : Need to sum over the possible assignments of the hidden variables.
  - Variable elimination
  - Separation

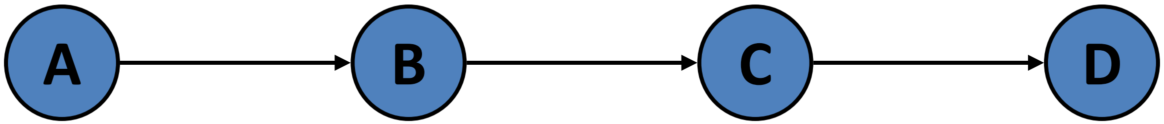
- Query variables: E_1
- Evidence variables: E_2
- The rest, E_3



P(W | Cloudy = True)

- E_1 = {W}
- E_2 = {Cloudy=True}
- E_3 = {Sprinkler, Rain}

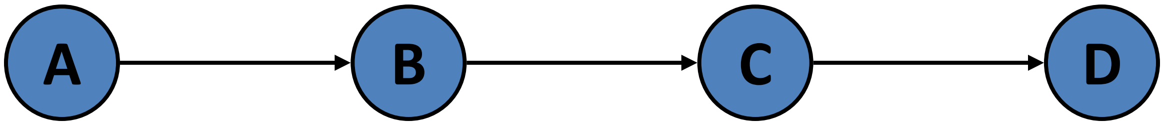
Inference: A Simple Case



- Suppose that we want to compute

P(D = d) from this network.

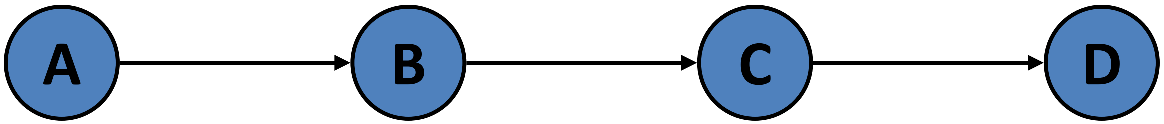
A Simple Case



- Compute P(D = d) by summing the joint probability over all possible values of the remaining variables A, B, and C:

$$P(D=d)=\sum_{a,b,c} P(A=a,B=b,C=c,D=d)$$

A Simple Case

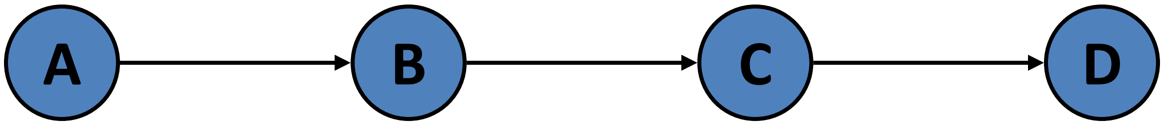


- Decompose the joint by using the fact that it is the product of terms of the form:

P(X | Parents(X))

$$P(D=d)=\sum_{a,b,c} P(D=d \mid C=c)P(C=c \mid B=b)P(B=b \mid A=a)P(A=a)$$

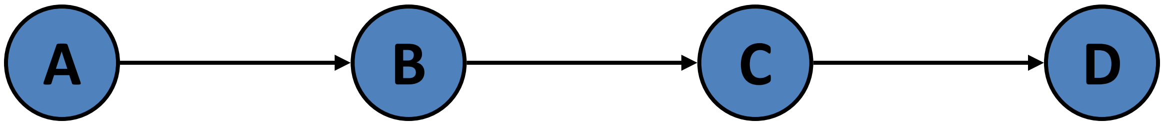
A Simple Case



- We can avoid computing the sum for all possible triplets (A,B,C) by distributing the sums inside the product

$$P(D=d)=\sum_{c}P(D=d\mid C=c)\sum_{b}P(C=c\mid B=b)\sum_{a}P(B=b\mid A=a)P(A=a)$$

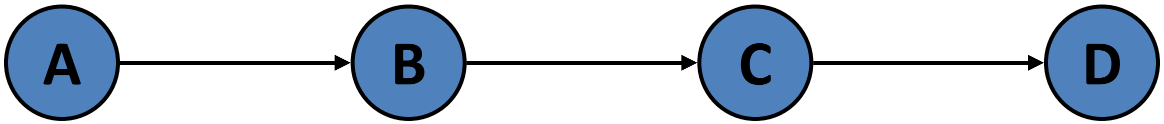
A Simple Case



This term depends only on B and can be written as a function $f_A(b)$

$$P(D=d)=\sum_{c} P(D=d \mid C=c)\sum_{b} P(C=c \mid B=b)\boxed{\sum_{a} P(B=b \mid A=a)P(A=a)}$$

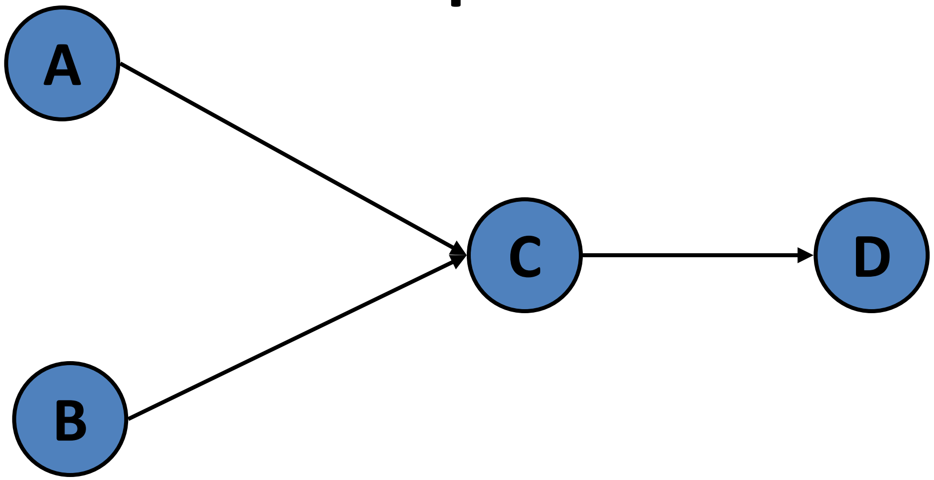
A Simple Case



This term depends only on c and can be written as a function $f_B(c)$

$$P(D=d)=\sum_{c} P(D=d \mid C=c) \boxed{\sum_{b} P(C=c \mid B=b) f_A(b)}$$

Example



$$P(D=d)=\sum_{a,b,c} P(D=d \mid C=c)P(C=c \mid B=b, A=a)P(B=b)P(A=a)$$

$$=\sum_{c} P(D=d \mid C=c)\sum_{b} P(B=b)\sum_{a} P(C=c \mid B=b, A=a)P(A=a)$$

$$=\sum_{c} P(D=d \mid C=c)\sum_{b} P(B=b)\sum_{a} f_1(a,b,c)$$

$$=\sum_{c} P(D=d \mid C=c)\sum_{b} f_2(b,c)$$

General Case: Variable Elimination

- Write the desired probability as a sum over all the unassigned variables

$$P(D=d)=\sum_{a,b,c} P(A=a, B=b, C=c, D=d)$$

- Distribute the sums inside the expression
  - Pick a variable
  - Group together all the terms that contain this variable

$$P(D=d)=\sum_{c} P(D=d \mid C=c)\sum_{b} P(C=c \mid B=b)\boxed{\sum_{a} P(B=b \mid A=a)P(A=a)}$$

  - Represent as a single function of the variables appearing in the group

$$P(D=d)=\sum_{c} P(D=d \mid C=c)\sum_{b} P(C=c \mid B=b) f_A(b)$$

  - Repeat until no more variables are left

General Case: Variable Elimination

- Write the desired probability as a sum over all the unassigned variables

Computation exponential in the size of the largest group → The order in which the variables are selected is important.

on

- – Pic e
- – Group ner all the terms that contain this variable

$$P(D=d)=\sum_{c} P(D \quad d \mid C=c)\sum_{b} P(C=c \mid B=b)\boxed{\sum_{a} P(B=b \mid A=a)P(A=a)}$$

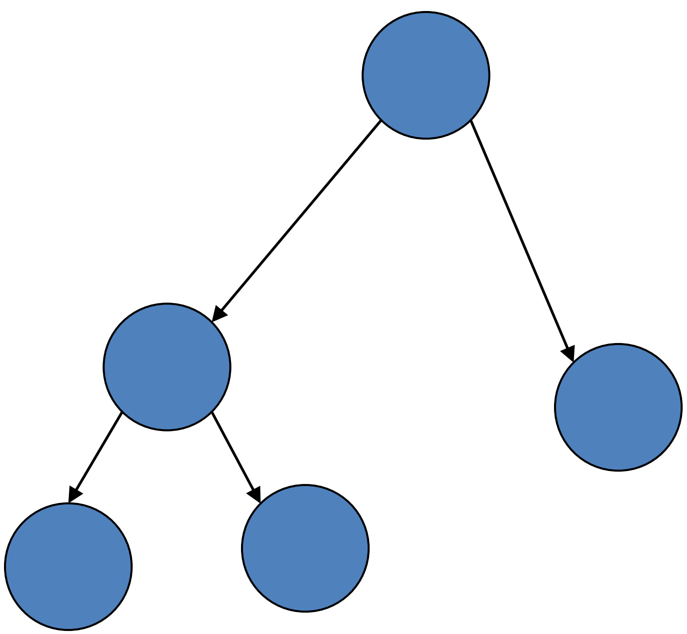
- – Represent as a single function of the variables appearing in the group

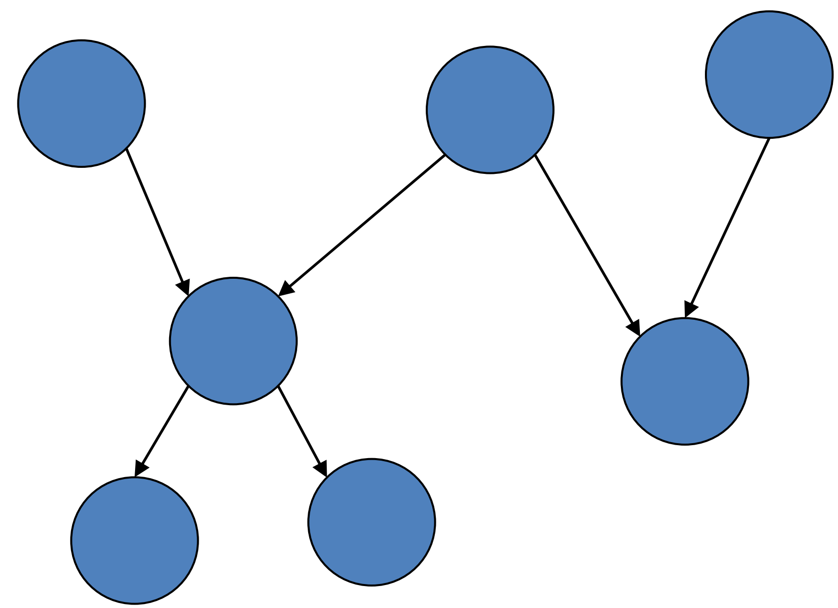
$$P(D=d)=\sum_{c} P(D=d \mid C=c)\sum_{b} P(C=c \mid B=b)f_{A}(b)$$

- – Repeat until no more variables are left

Special Case

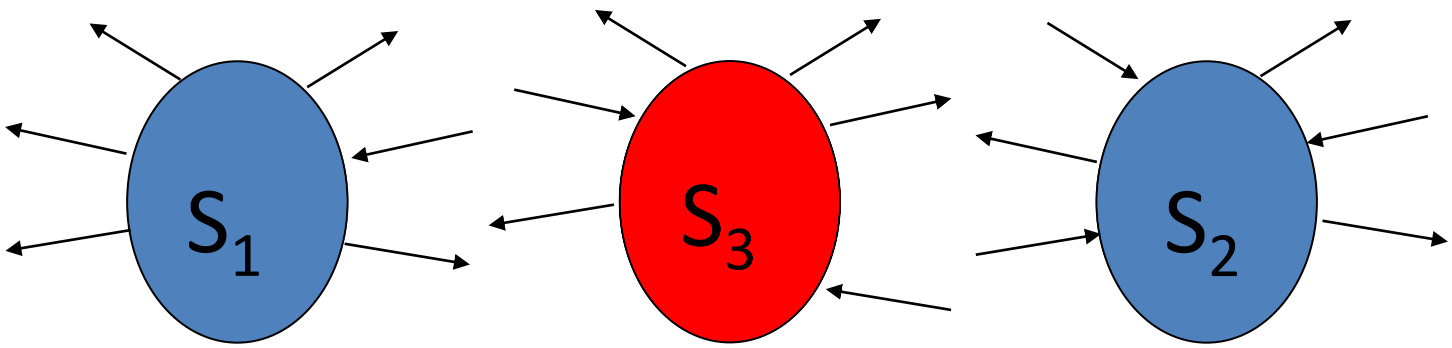
- Polytrees: Undirected version of the graph is a tree = there is a single undirected path between two nodes
- In this case: Inference linear in the number of nodes ($d^{k+1}n$)
- General case: See later approximate inference (e.g., sampling)





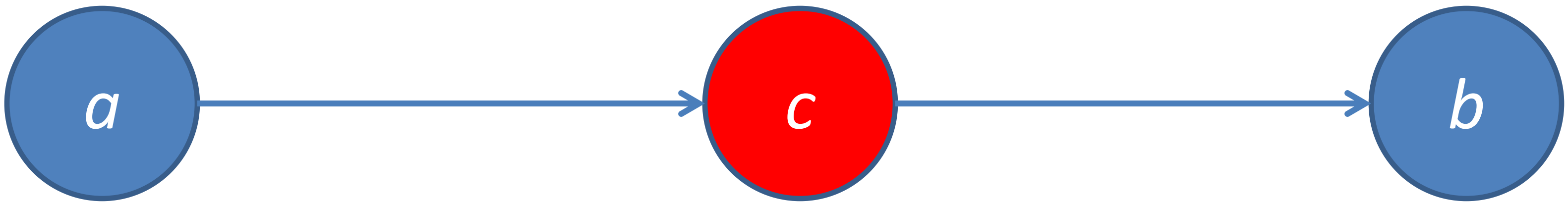
Conditional independence

- P(any assignments to S_1 | any assignments to S_2 ,any assignments to S_3) = P(assignment to S_1 | assignments to S_3)
- P(any assignments to $S_{1,}$ any assignments to S_2 |any assignments to S_3) = P(assignment to S_1)P(assignments to S_2)

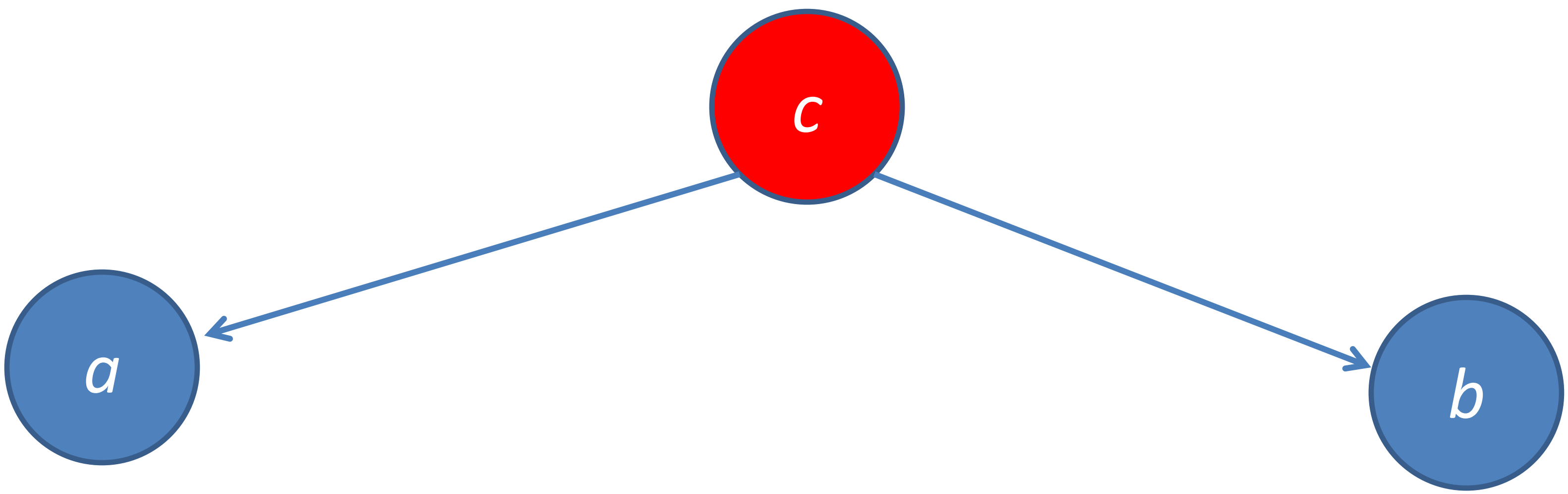


Finding independences

- The more *in*dependence relations we can find, the faster the inference → Test to find independences?



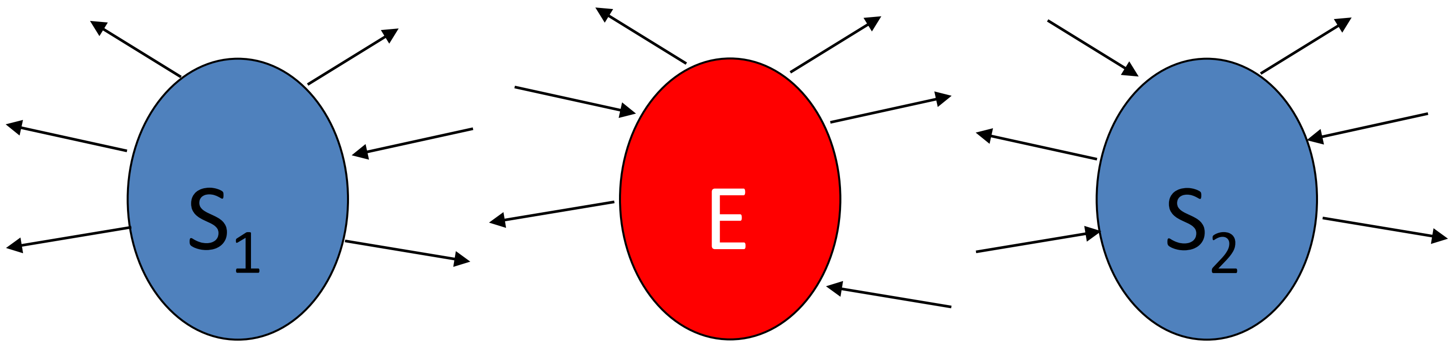
$p(b|c,a) = P(b|c) \qquad a \perp b \mid c$



$p(a,b|c) = P(a|c)\,P(b|c) \qquad a \perp b \mid c$

More General

- How can we find if S_1 and S_2 are conditionally independent given E?

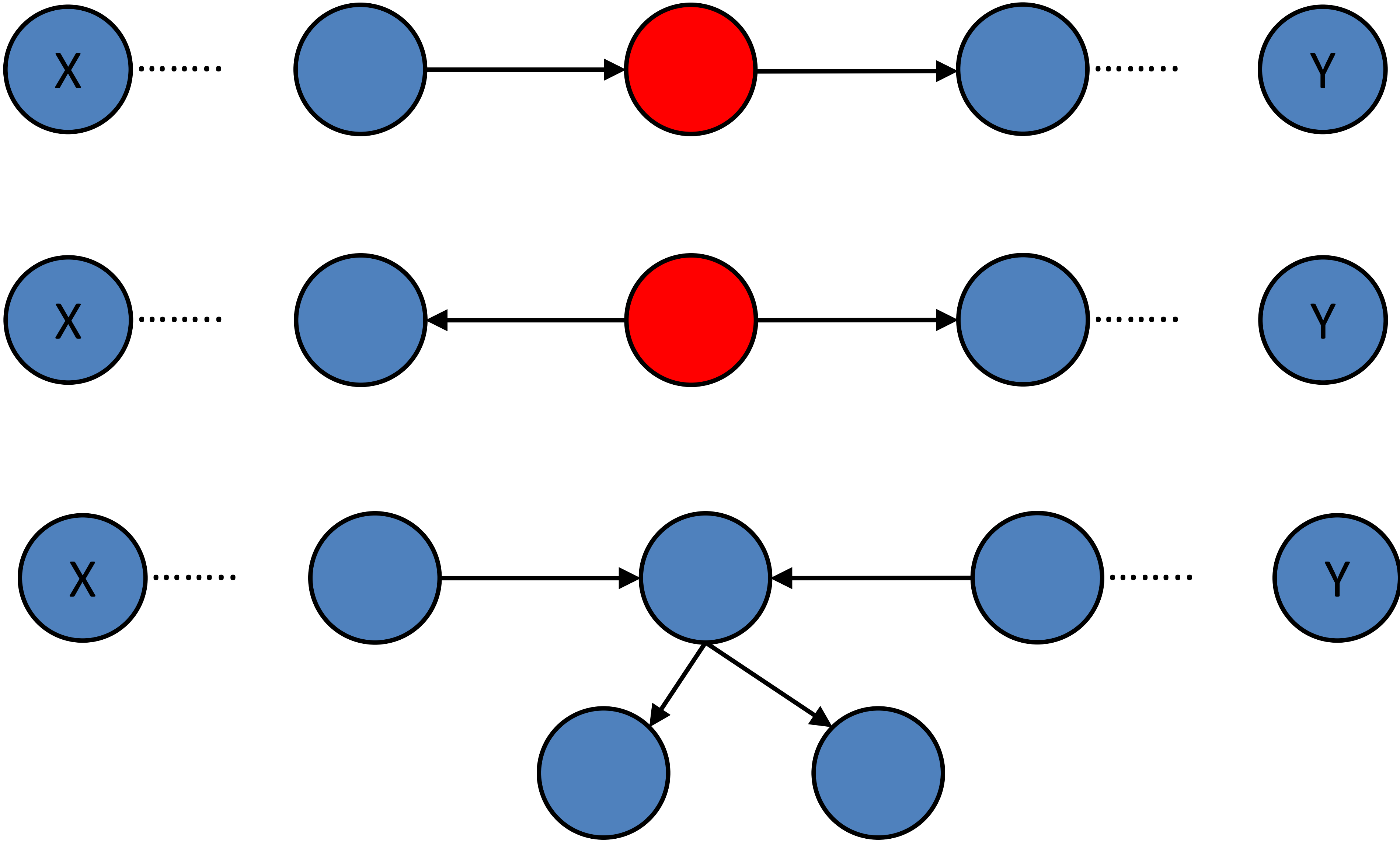


P (assignments to S_1 | E and assignments to S_2) = P (assignments to S_1 | E)

- Why is it important and useful?

We can simplify any computation that contains something like $P(S_1 \mid E , S_2)$ by $P(S_1 \mid E)$

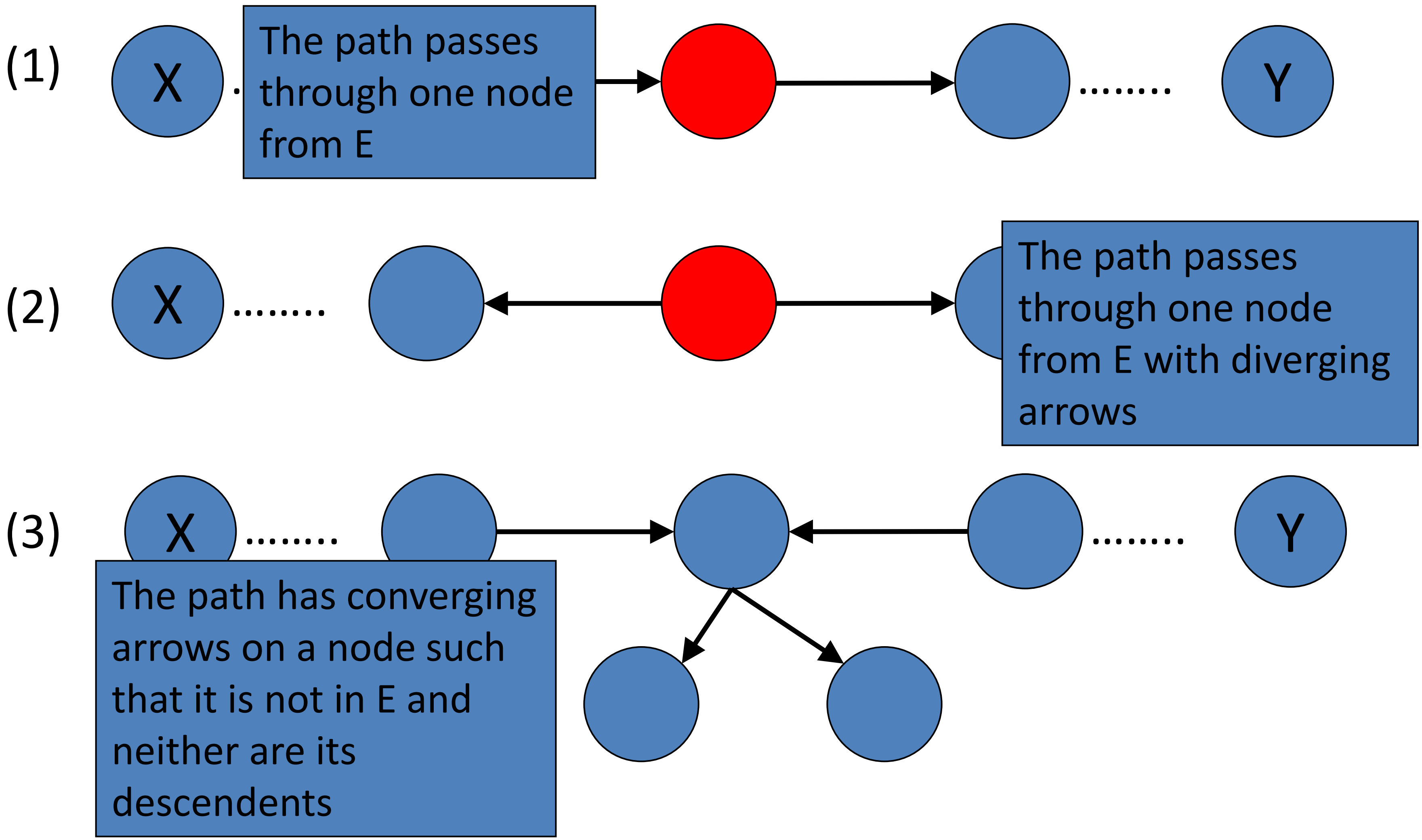
Intuitively E stands in between or "blocks" S_1 from S_2



X
Y
X
Y
X
Y

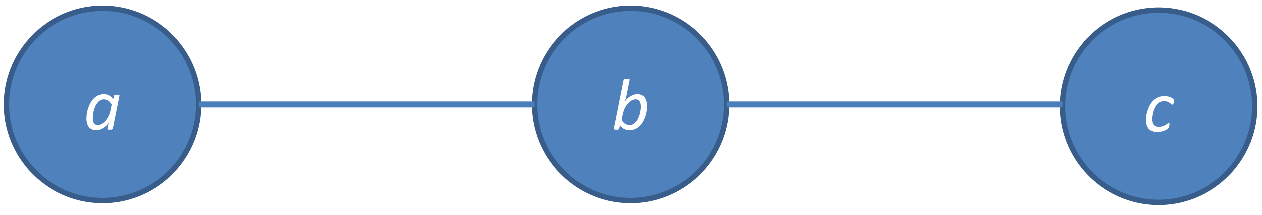
Blockage: Formal Definition

- A path from a node X to a node Y is *blocked* by a set E if either:

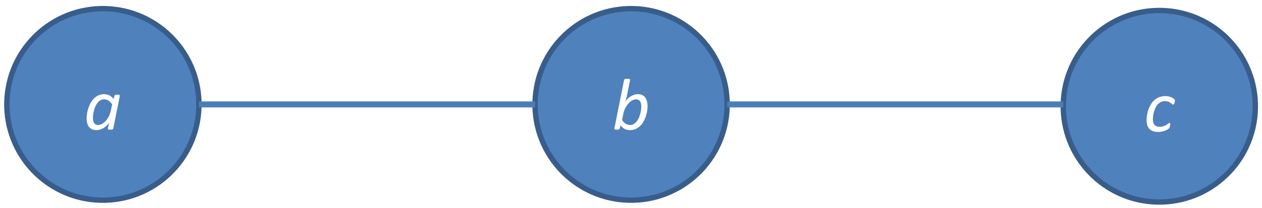


General case: Undirected

$$P(a, b) = \varphi(a, b) \qquad P(a, b) = \varphi_1(a)\varphi_2(b)$$

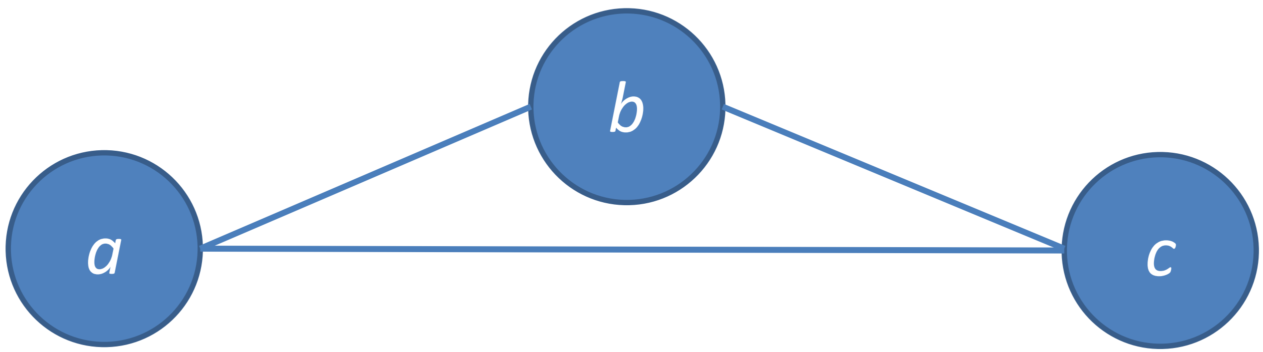


$$P(a,b,c) = \varphi_1(a,b)\,\varphi_2(b,c)$$



$$P(a, b, c) = \varphi_1(a, b)\,\varphi_2(b, c)$$

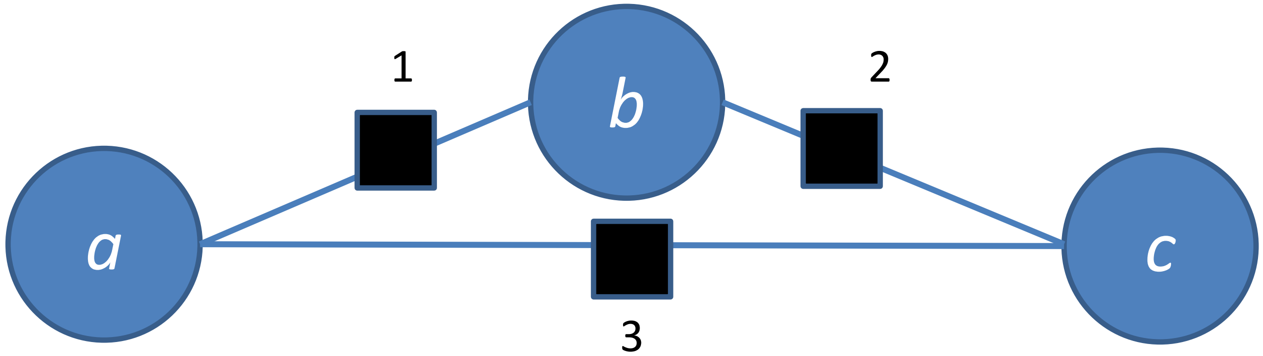
$a \perp c \mid b$ because all paths between a and c go through b



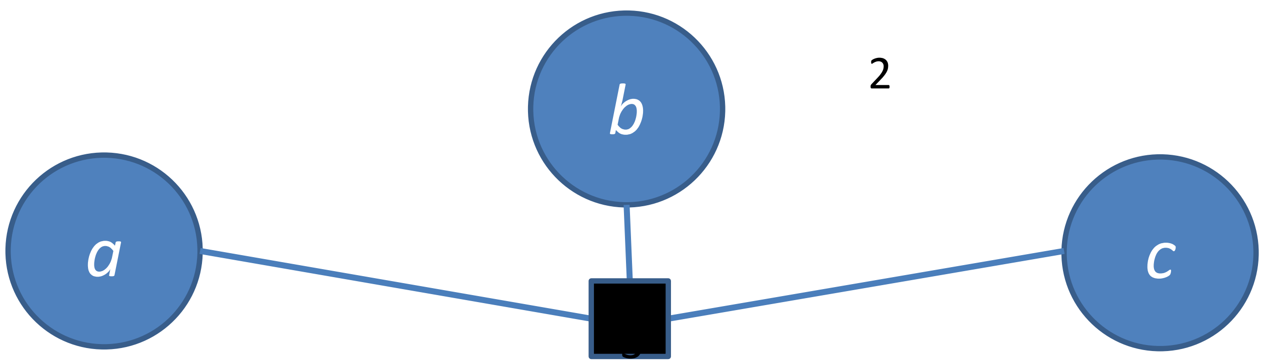
$$P(a,b,c) = \varphi_1(a,b)\,\varphi_2(b,c)\,\varphi_3(a,c)$$

$$P(a,b,c) = \varphi\,(a,b,c)$$

Factor graphs

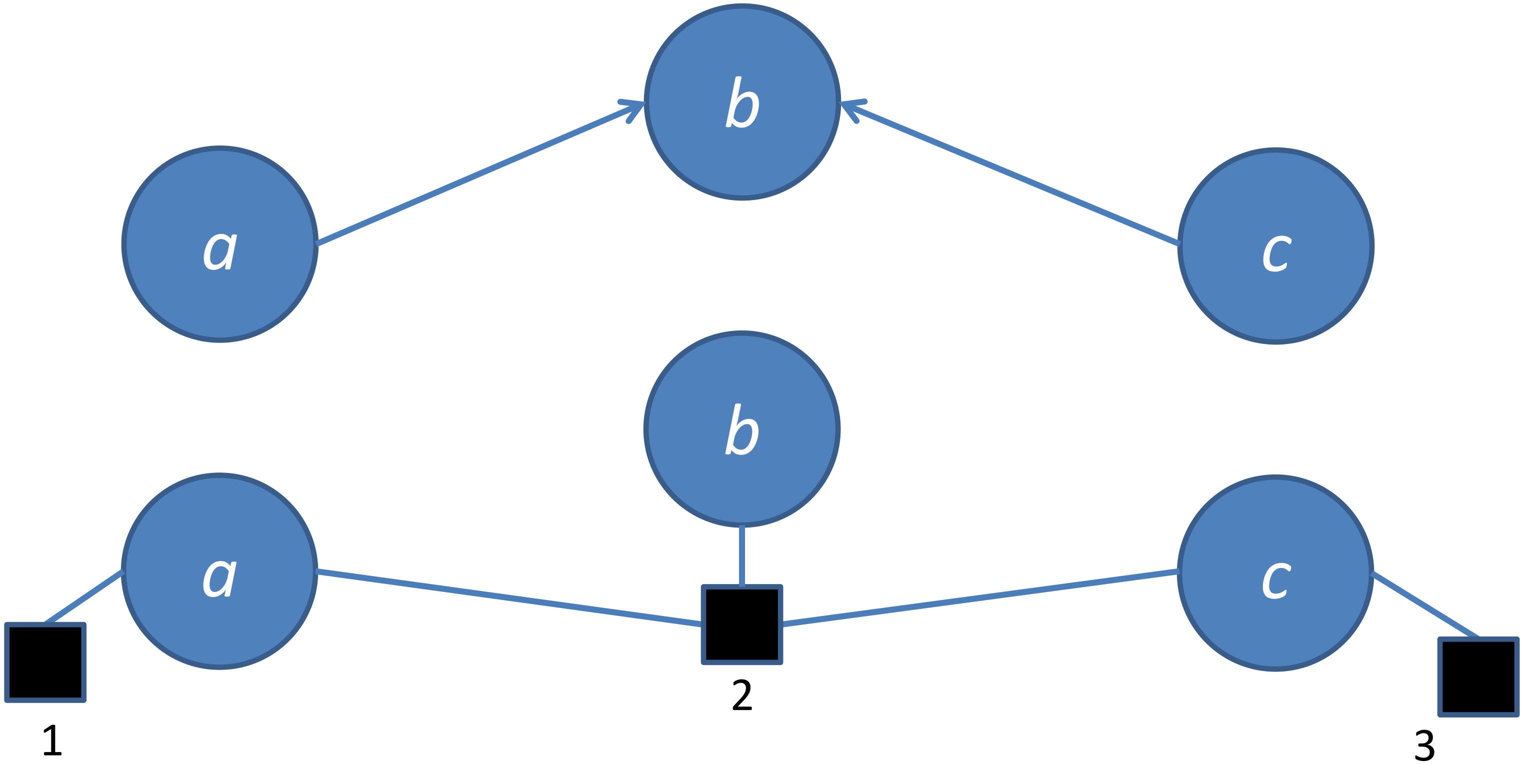


$$P(a,b,c) = \varphi_1(a,b)\,\varphi_2(b,c)\,\varphi_3(a,c)$$



$$P(a,b,c) = \varphi(a,b,c)$$

Directed vs. undirected



$$P(a,b,c) = \varphi_1(a,b)\,\varphi_2(b,c)\,\varphi_3(a,c)$$

$$\varphi_1(a) = P(a) \quad \varphi_3(c) = P(c)$$

$$\varphi_2(a,b,c) = P(b|a,c)$$

- N regions
- M possible labels
- Somehow, there is a way to estimate how likely a label is given image features $P(l_i|f)$
- We want to find the assignment of labels that optimizes $P(l_1, .., l_N|f)$

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- Everything is independent:

l_1 .... l_N

$$P(l_1, .., l_N | f) = \prod P(l_i | f)$$

Gives really stupid results because it does not take into account the distribution of likely relative occurrence of the labels

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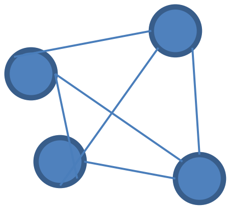
- Everything is dependent:



$$P(l_1,..,l_N|f)\alpha \prod P(f|l_i)P(l_1,..,l_N)$$

Hard to learn or represent $P(l_1,..,l_N)$

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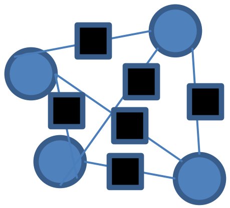
- Factor pairwise dependencies:



$$P(l_1, .., l_N) = \prod \varphi(l_i, l_j)$$

$\varphi(l_i, l_j)$ can be estimated from co-occurrence statistics from training data

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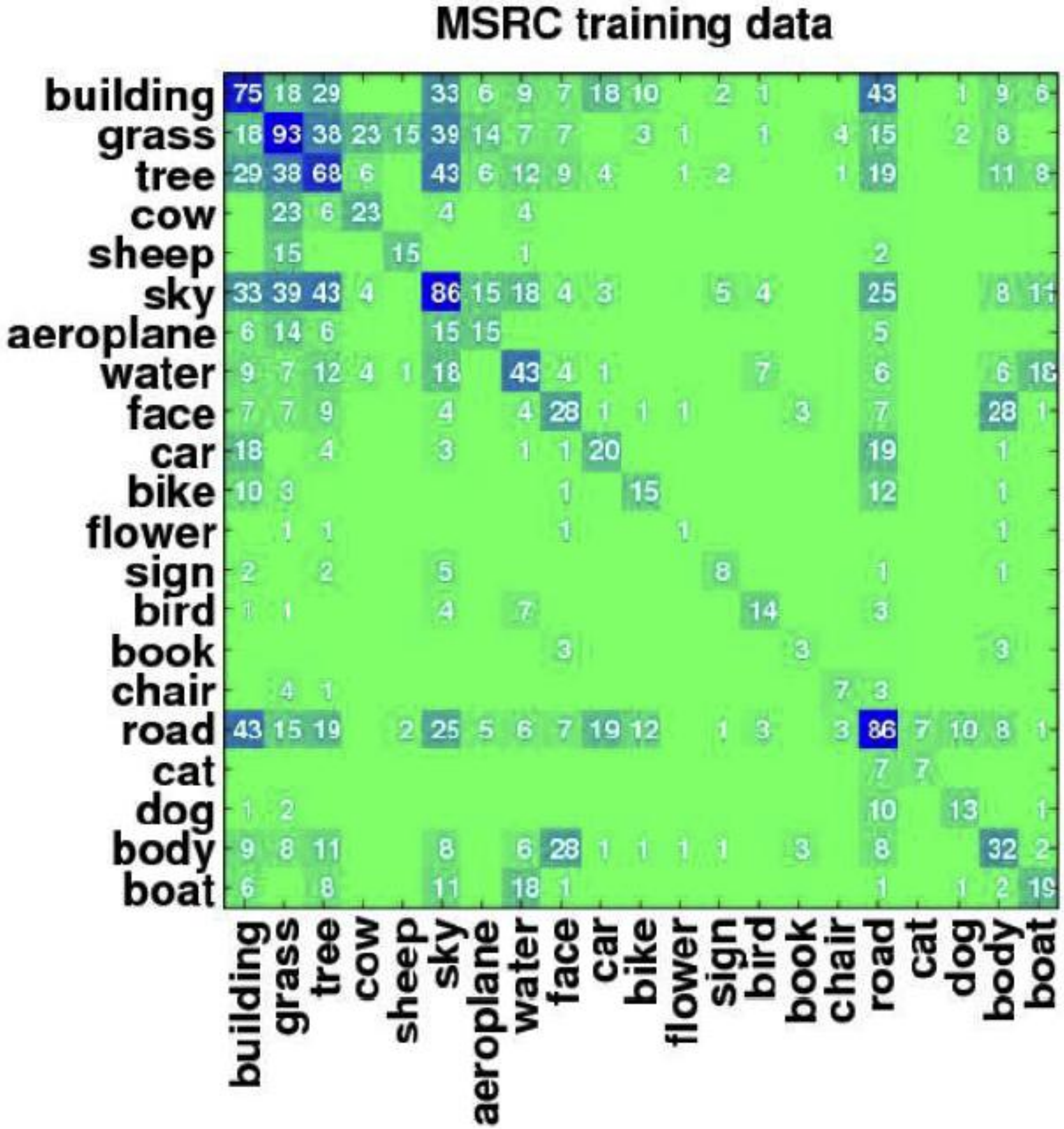

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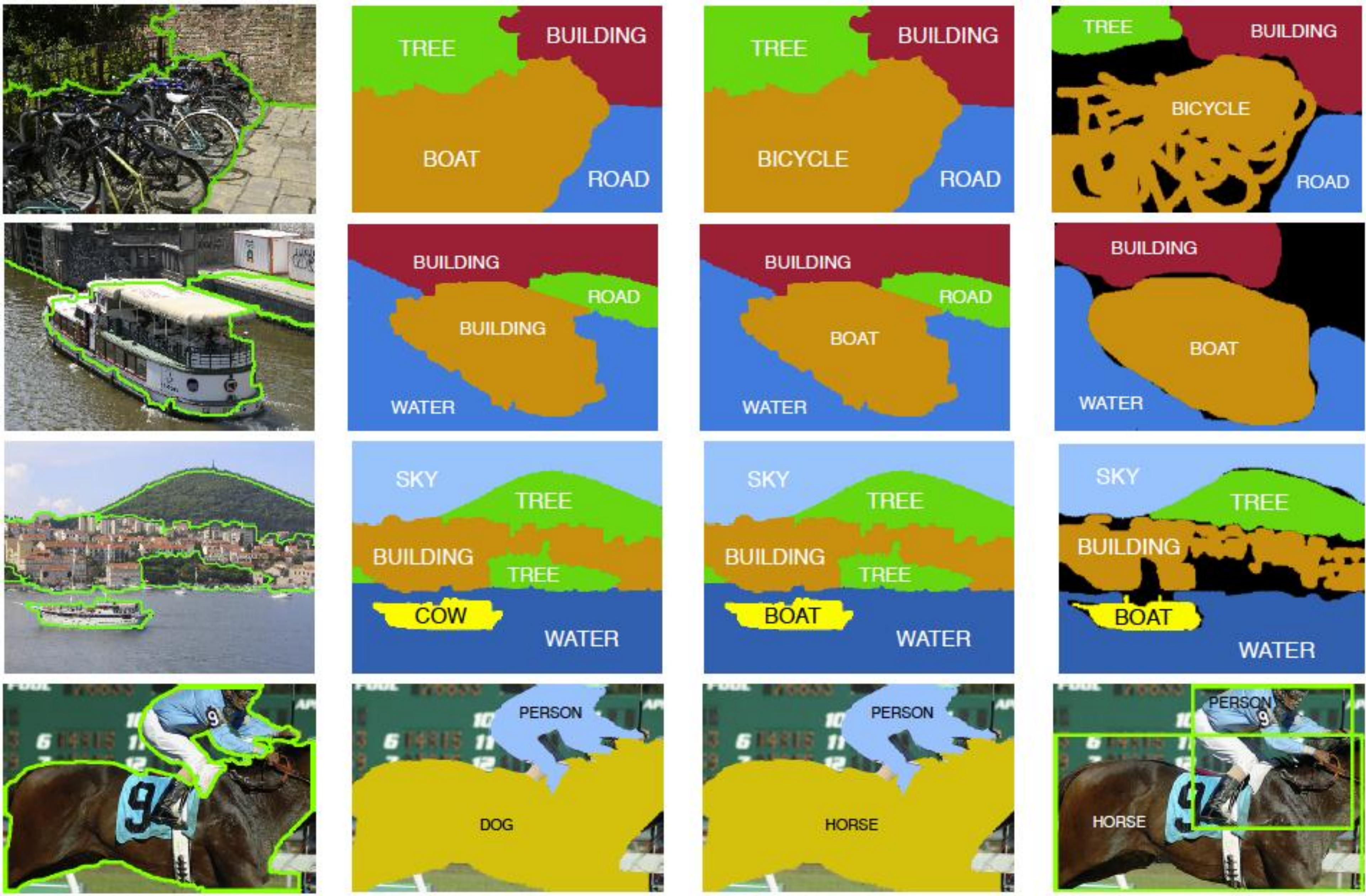

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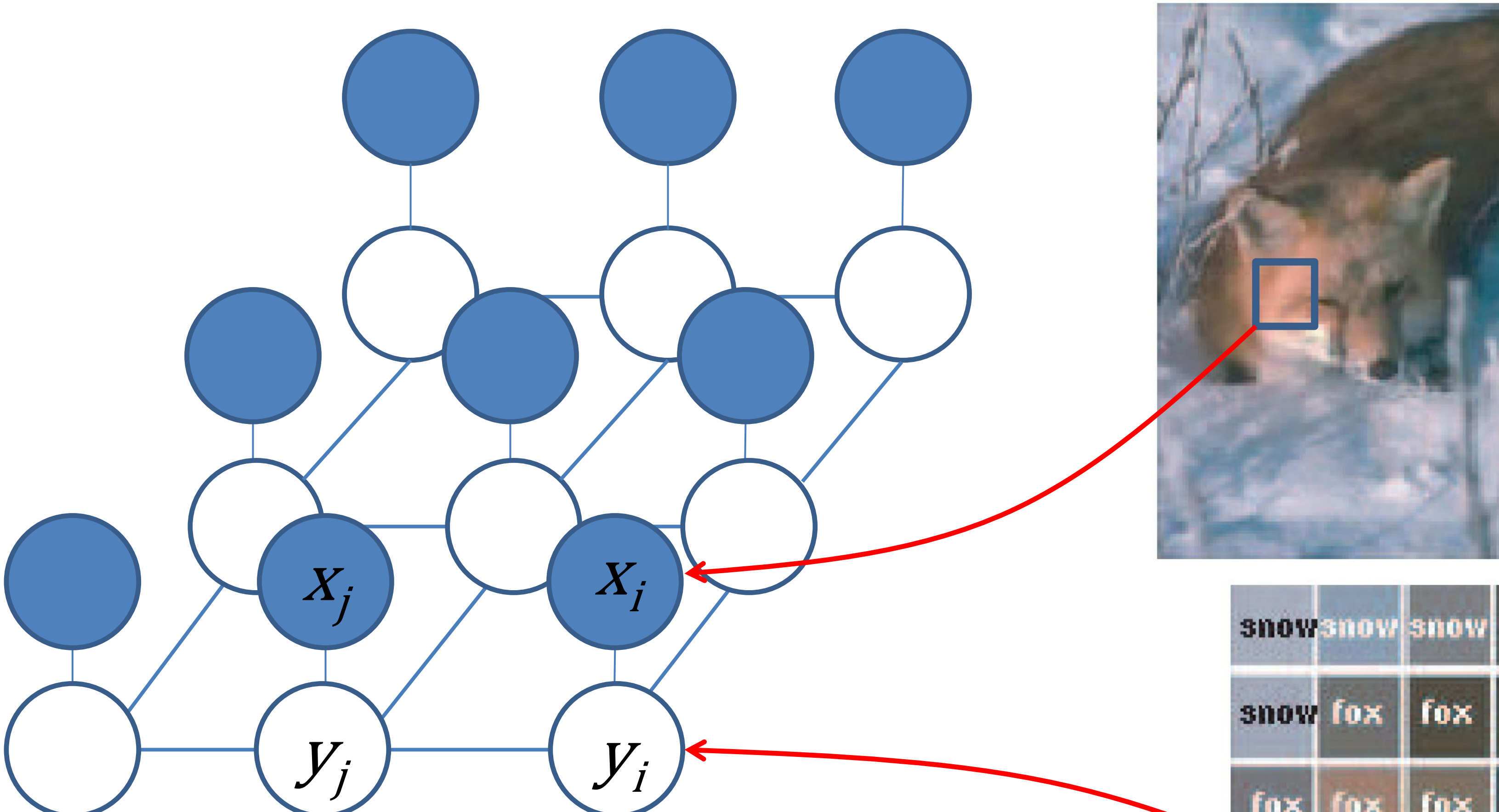
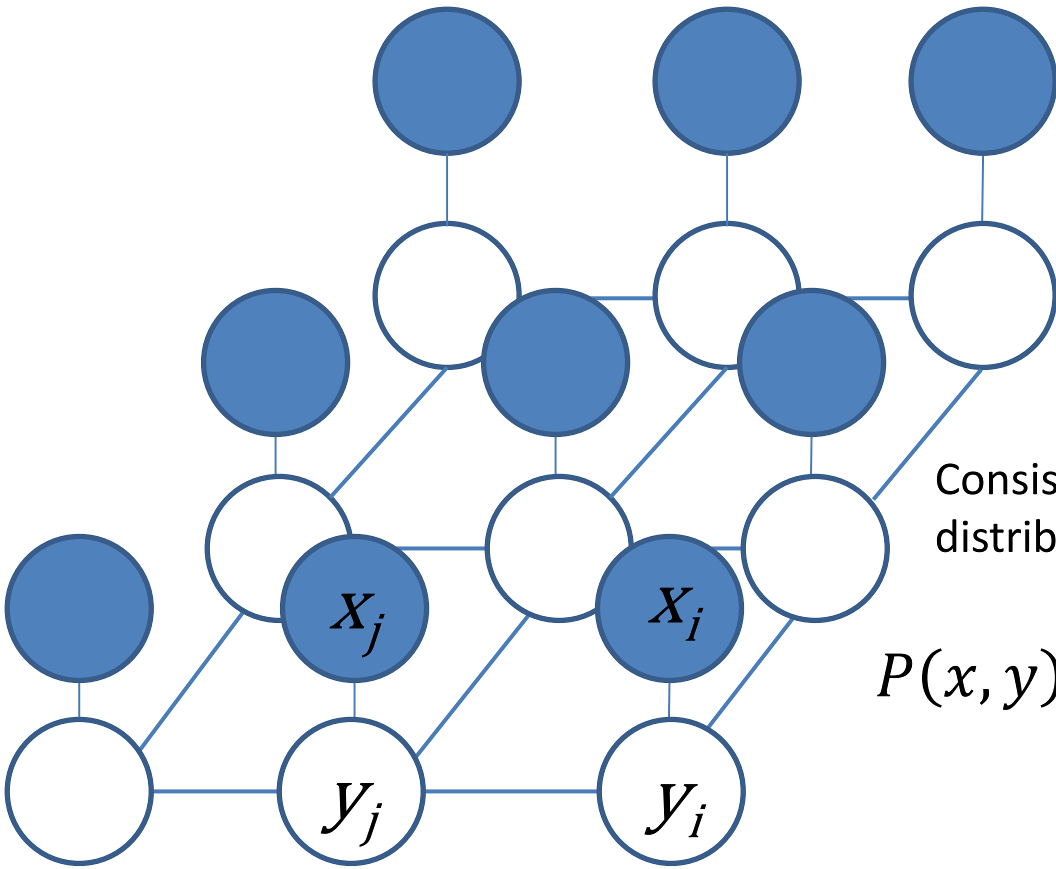
Example: MRF for image labeling



- y = class label (e.g., road, car, etc...)
- x = image data at local patch

Example from Carbonetto, de Freitas & Barnard, ECCV'04

Example: MRF for image labeling



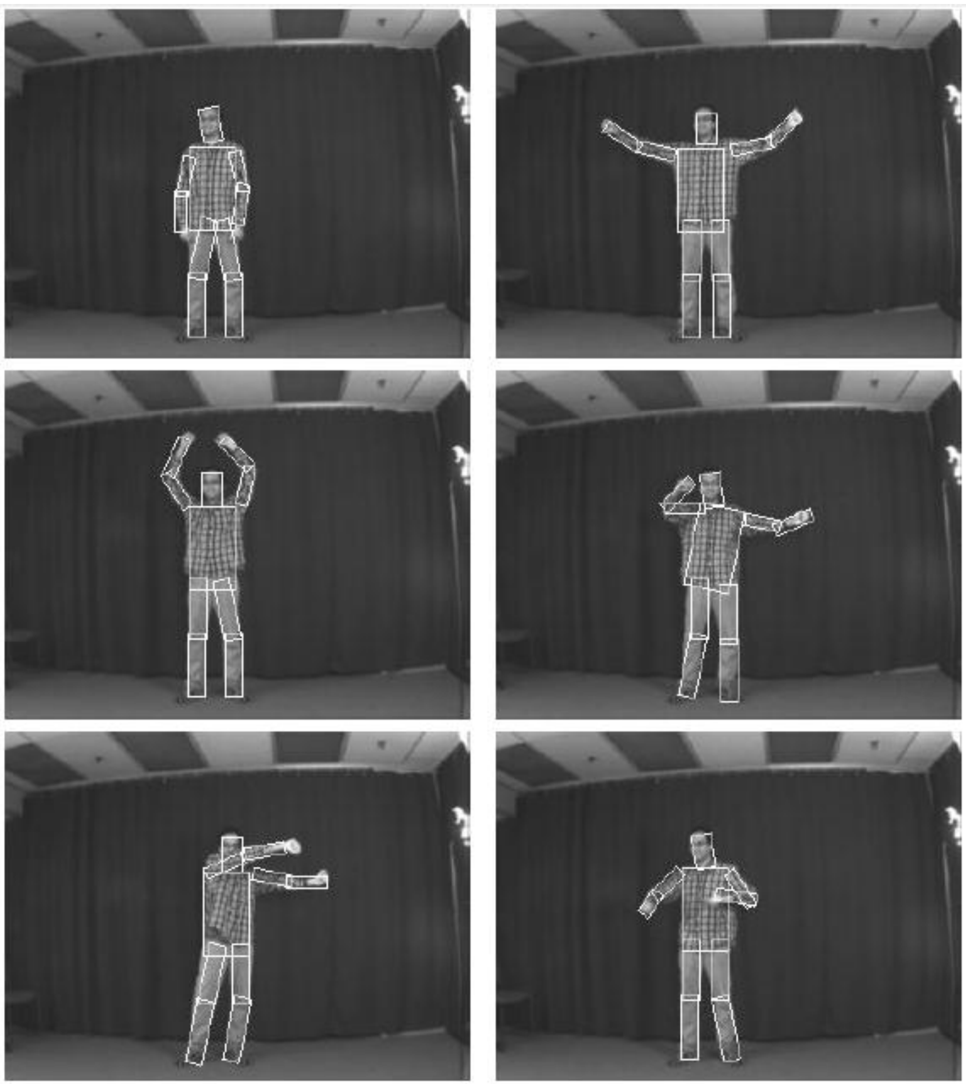
Consistency on spatial distribution of labels

Agreement of label vs. input data

$$P(x,y)\ \alpha \prod_{i,j} \varphi_{ij}(y_i, y_j) \prod_i \varphi_i(x_i, y_i)$$

- y = class label (e.g., road, car, etc…)
- x = image data at local patch

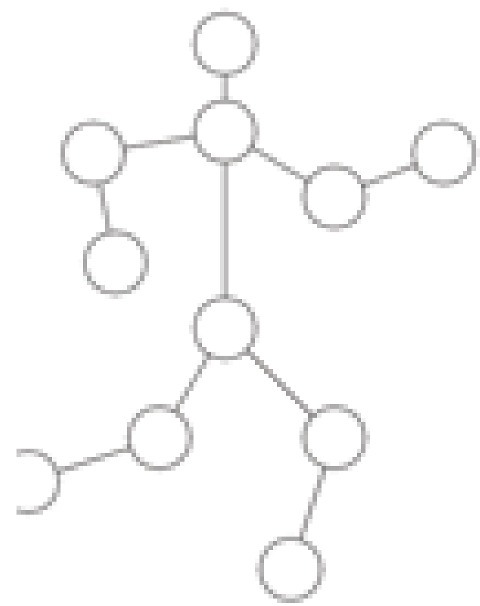
Example: Inferring human poses



Example from Felzenszwalb'04

x_i = Input image data at limb i

y_i = Pose (location and orientation) of limb i



- Note: Efficient because tree-structured