CMU MDPs 15-381/781

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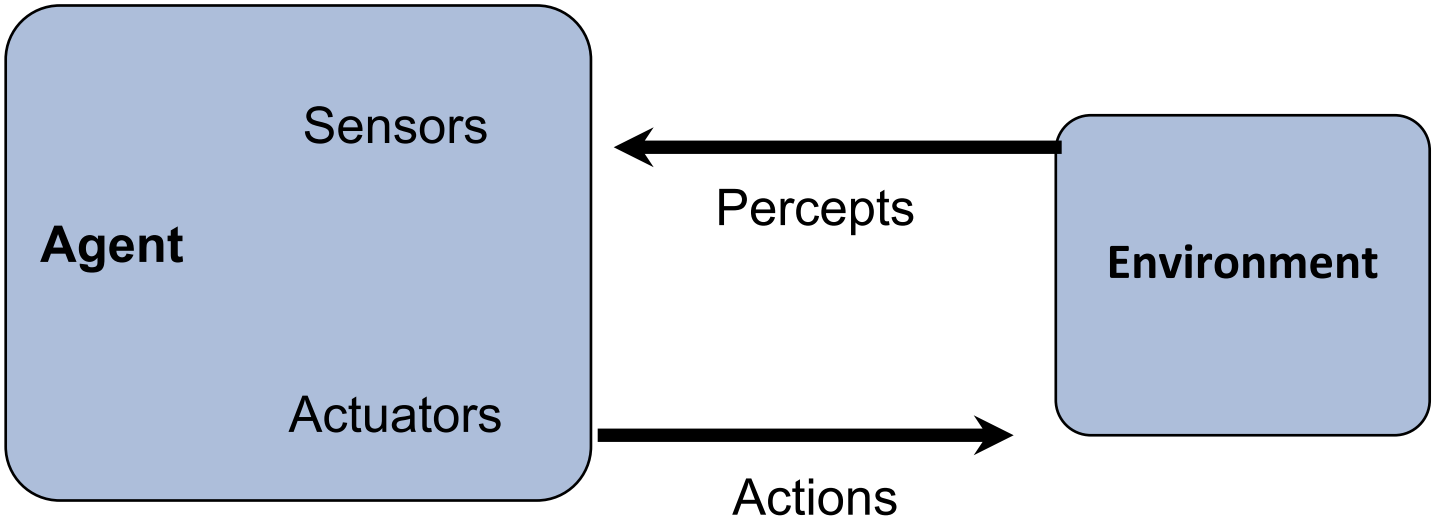
- DeepMind

So long certainty…



- Until now, result of taking an action in a state was deterministic

Reasoning Under Uncertainty

	Actions Don't Change State of the World	Actions Change State of the World
Learn model of outcomes	Multi-armed bandits	Reinforcement Learning
Given model of stochastic outcomes	Decision theory	Markov Decision Processes

Expectation

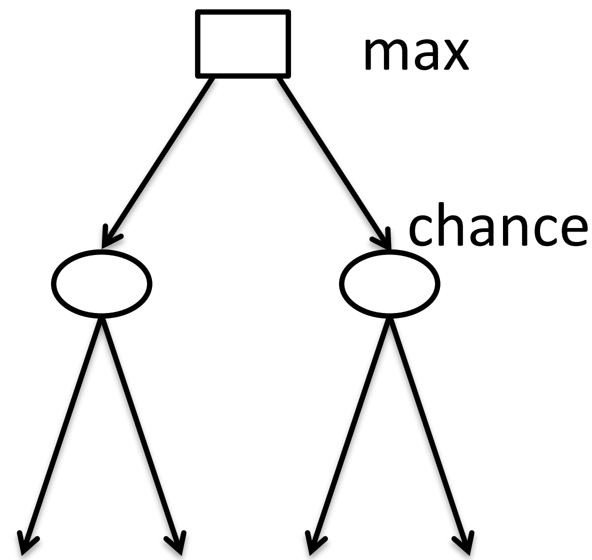
- The expected value of a function of a random variable is the average, weighted by the probability distribution over outcomes

- Example: expected time if take the bus
- Time: 5 min + 30 min
- Probability: 0.7 + 0.3

12.5 min

Where Do Probabilities Come from?

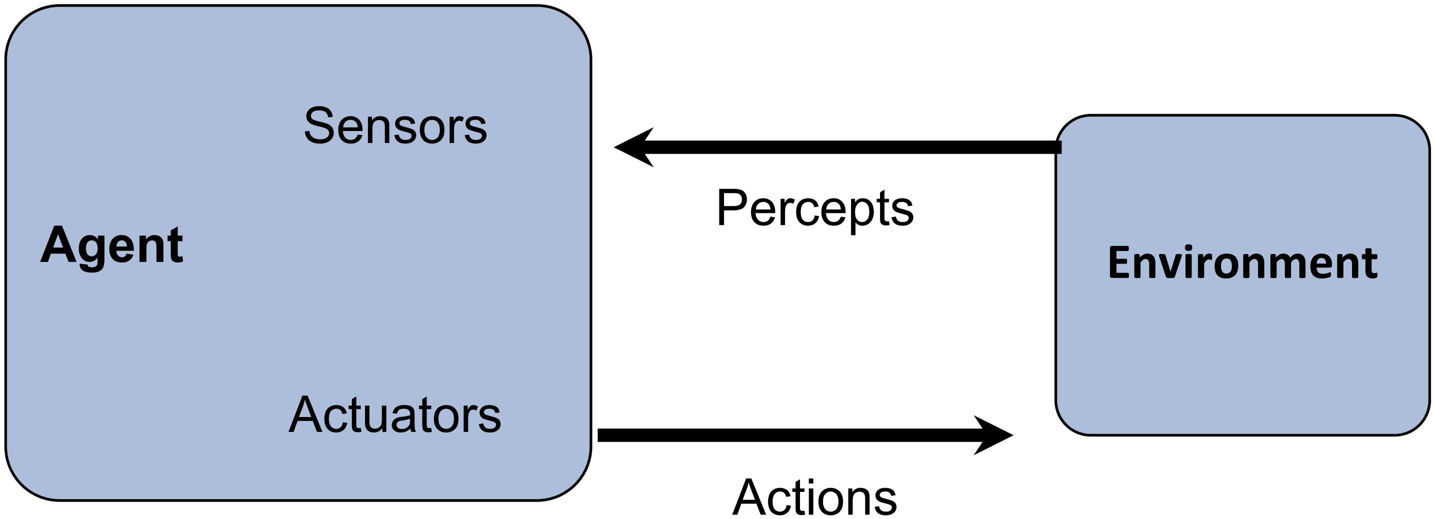
- Models
- Data
- For now assume we are given the probabilities for any chance node



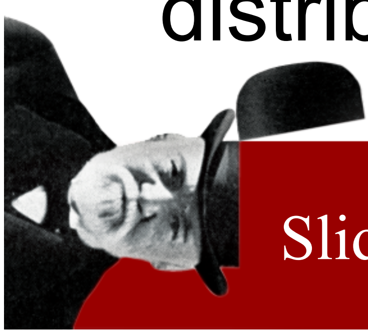
Reasoning Under Uncertainty

Given model of stochastic outcomes	Decision theory	**Markov Decision Processes**
	Actions Don’t Change State of the World	Actions Change State of the World

(Stochastically) Change the World

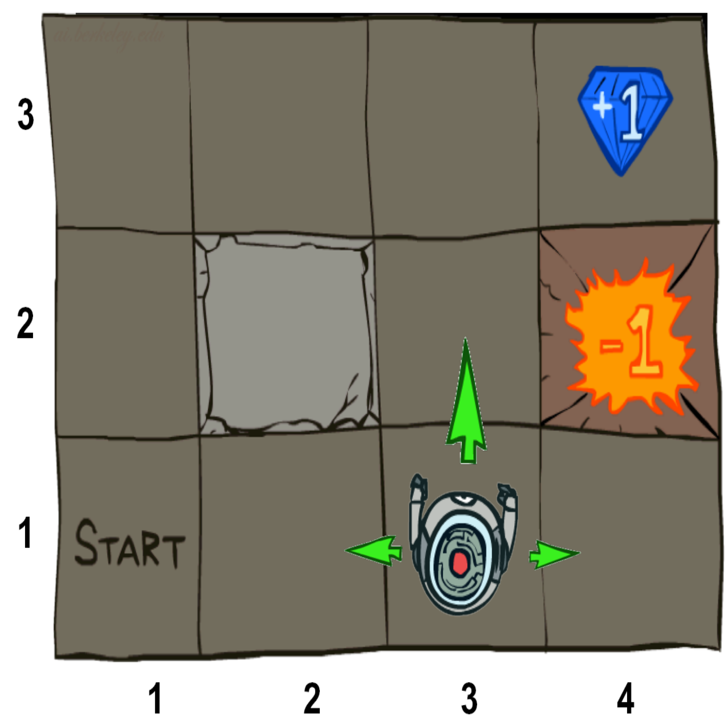


- Like planning/search, actions impact world
- But exact impact is stochastic: probability distribution over next states



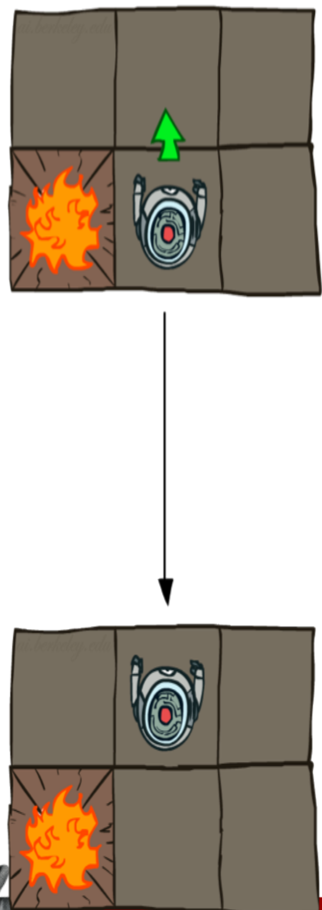
Example: Grid World

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent’s path
- The agent receives rewards each time step
  - Small “living” reward each step (can be negative)
  - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards
- Noisy movement: actions do not always go as planned
  - 80% of the time, action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have gone, agent stays put

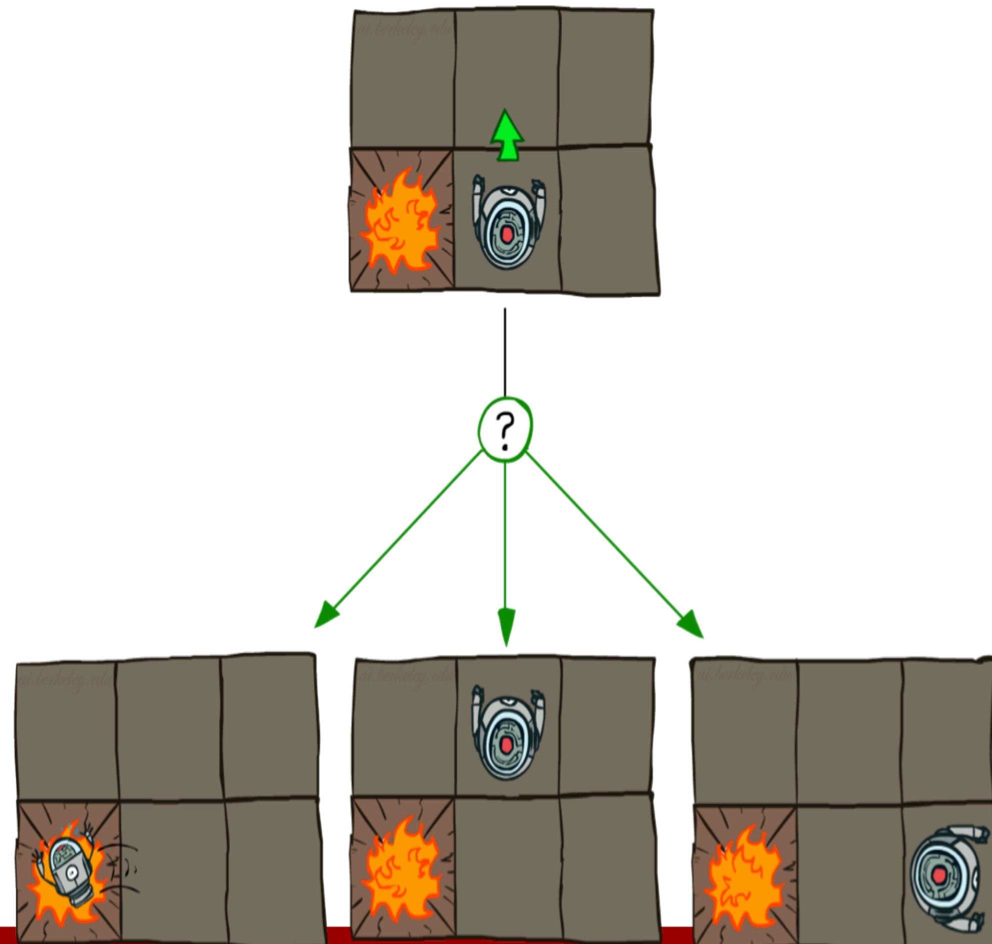


Grid World Actions

Deterministic Grid World

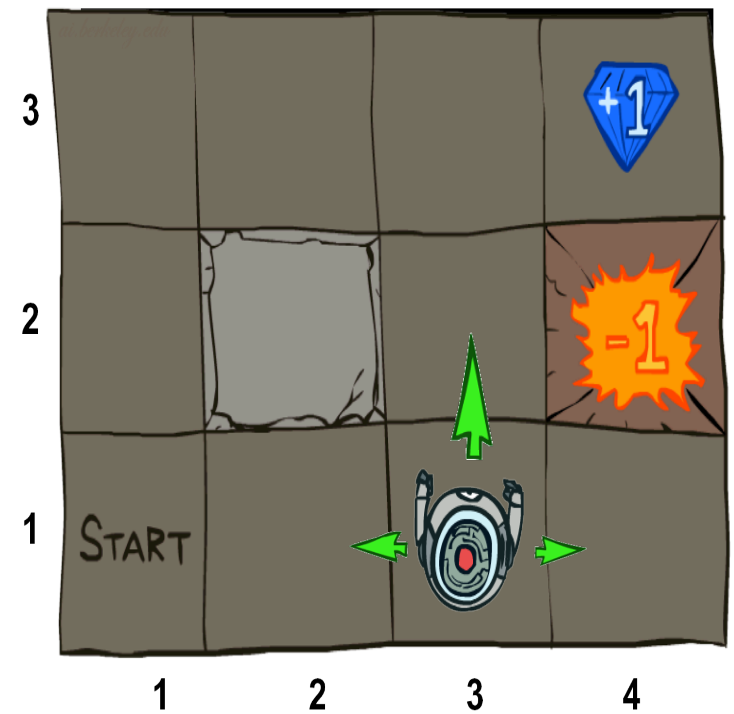


Stochastic Grid World



Markov Decision Processes

- Set of states $s \in S$
- Set of actions $a \in A$
- Transition func. T(s, a, s')
  - Probability that a from s leads to s', i.e., P(s'| s, a)
- Reward func. R(s, a, s') / R(s) / R(s,a)
- Start state or states (could be all S)
- Maybe a terminal state
- Discount factor
- MDPs are non-deterministic search problems



Markov Decision Processes

Markov Property

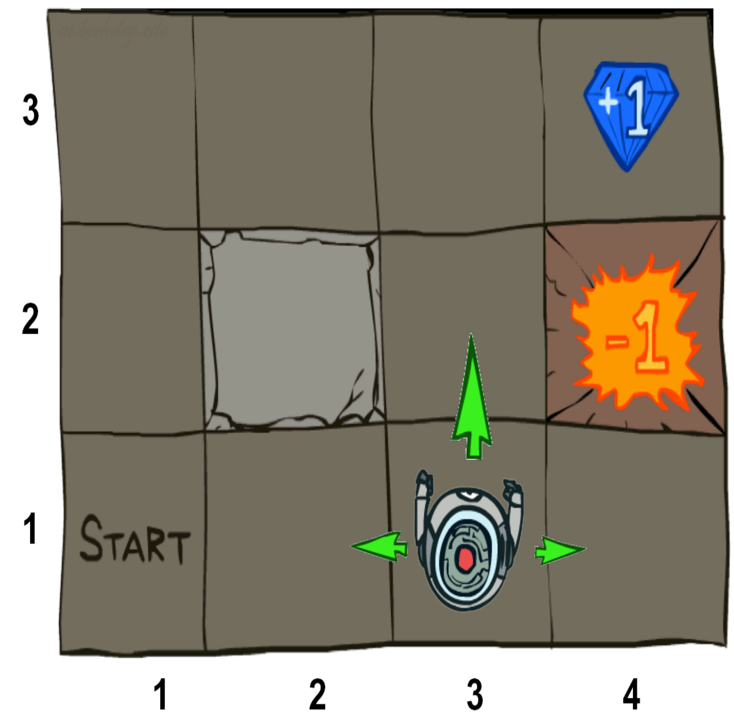
- Called **Markov** decision process because the outcome of an action depends only on the current state
- $p(s_{t+1}|s_1,a_1,s_2,a_2,\ldots s_t,a_t)=p(s_{t+1}|s_t,a_t)$

Policies

- In deterministic single-agent search problems, we wanted an optimal plan, or sequence of actions, from start to a goal
- In MDPs instead of plans, we have a policies
- A policy π^*: S $\rightarrow$ A
  - Specifies what action to take in each state

How Many Policies?

- How many non-terminal states?
- How many actions?
- How many deterministic policies over non-terminal states?



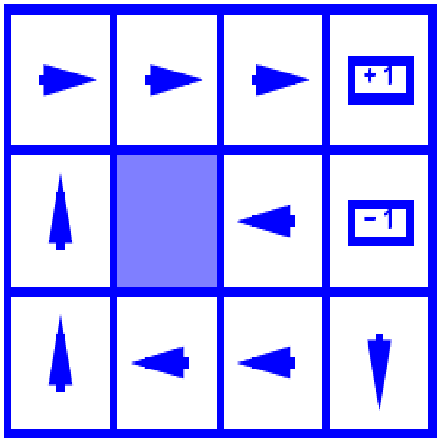
Optimal Policies

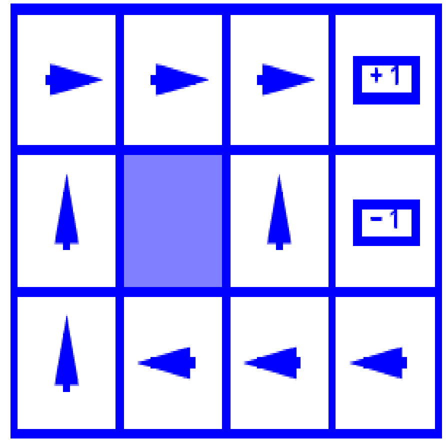
- Optimal plan had minimal cost to reach goal
- Utility or value of a policy π starting in state s is the expected sum of future rewards will receive by following π starting in state s

- Optimal policy has maximal expected sum of rewards from following it

Optimal Policies

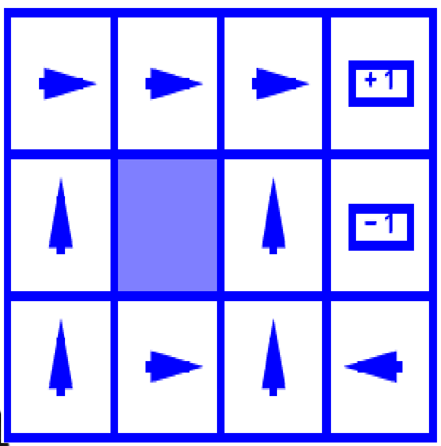
R(s) = -0.01

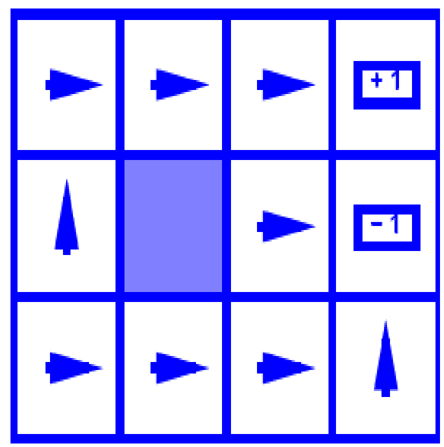




R(s) = -0.03

R(s) = -0.4

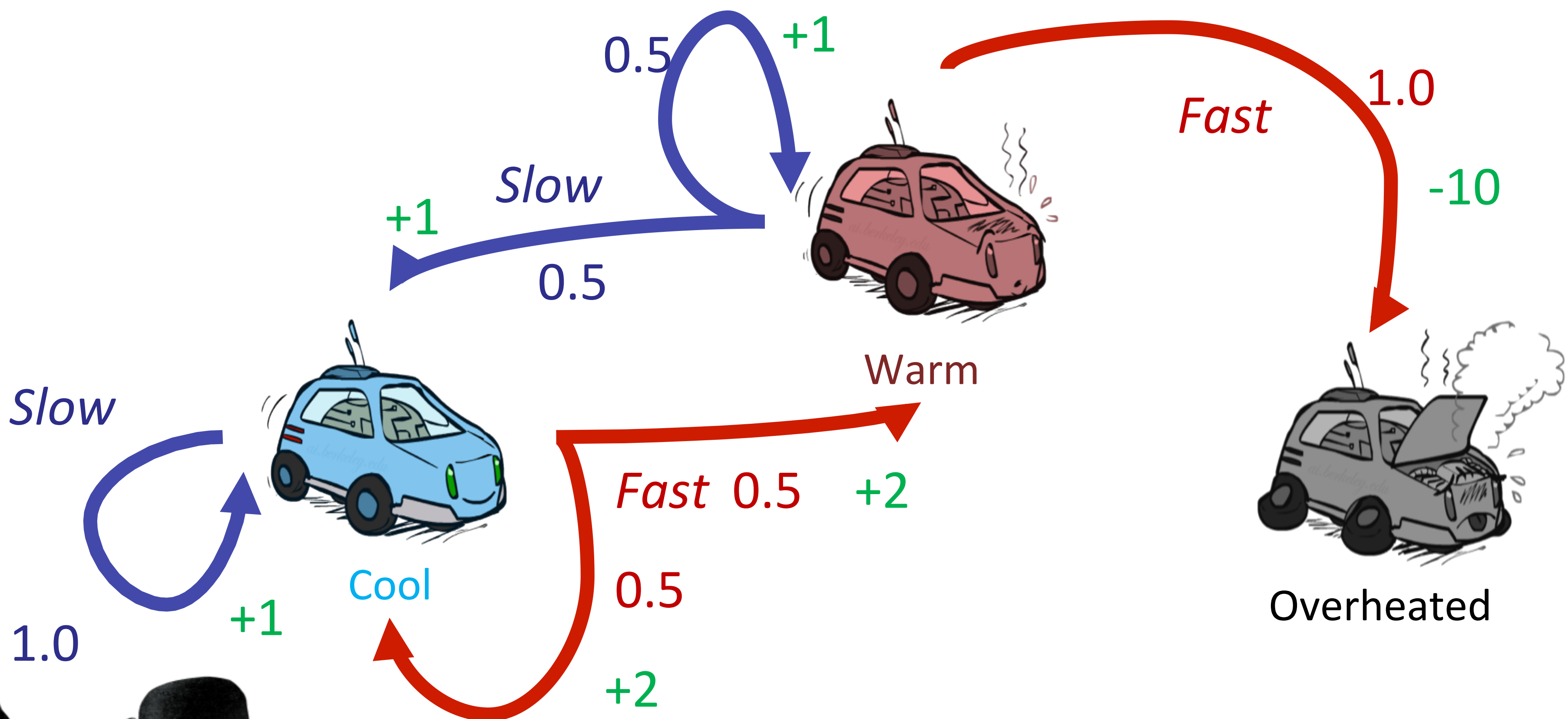




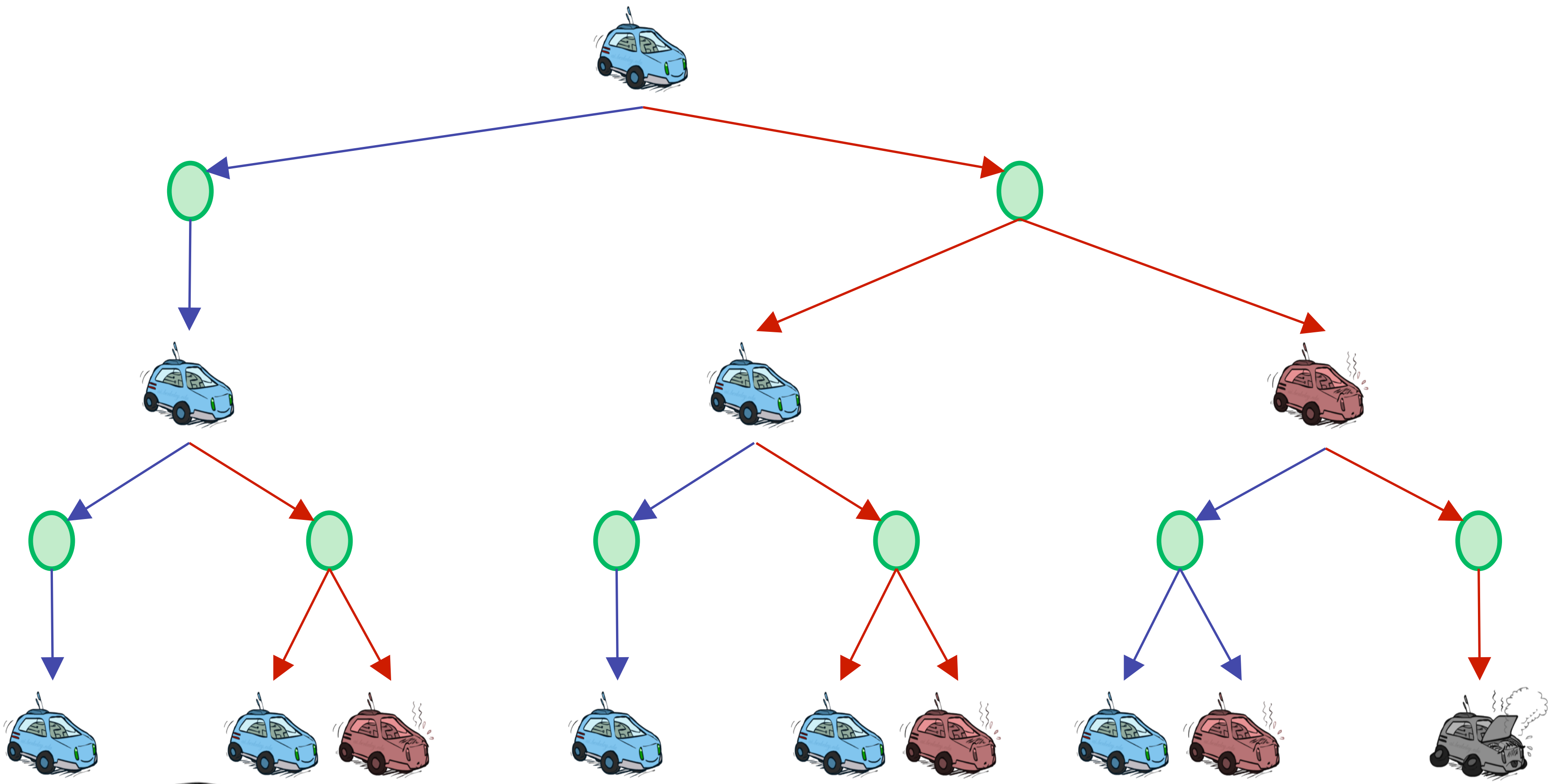
R(s) = -2.0

Example: Racing

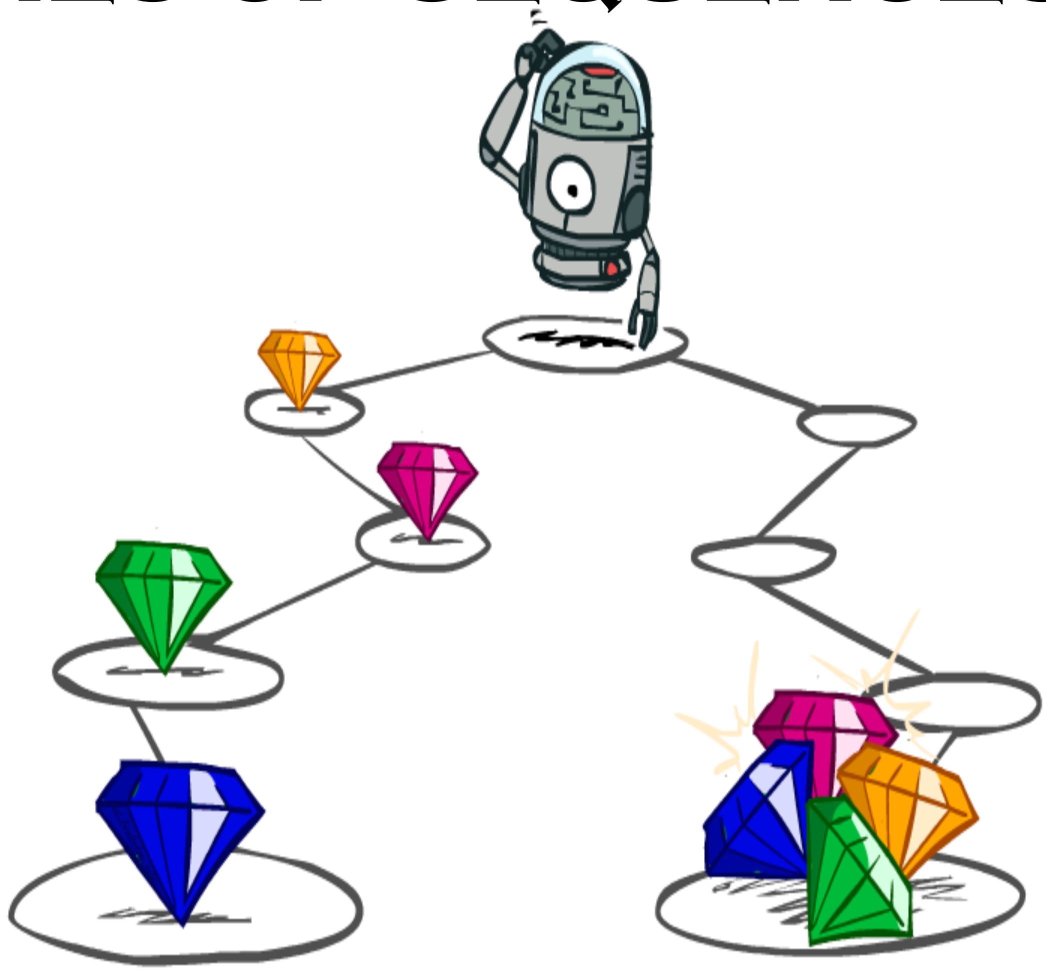
- A robot car wants to travel far, quickly
- Three states: Cool, Warm, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward



Racing Search Tree



Utilities of Sequences



Utilities of Sequences

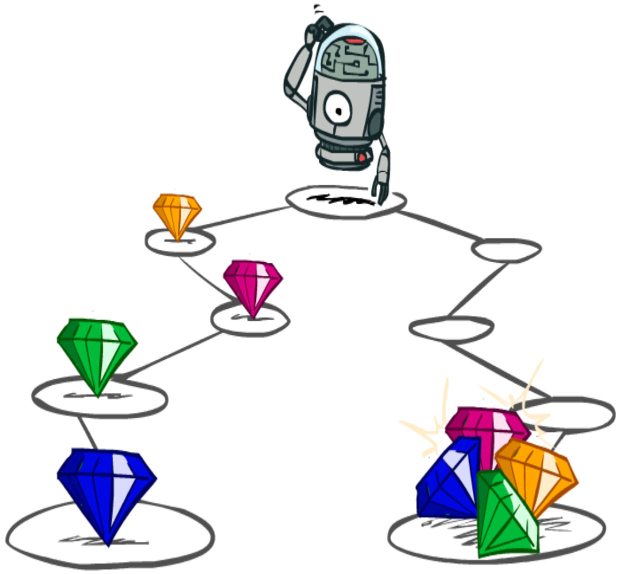
- What preferences should an agent have over reward sequences?

- More or less?

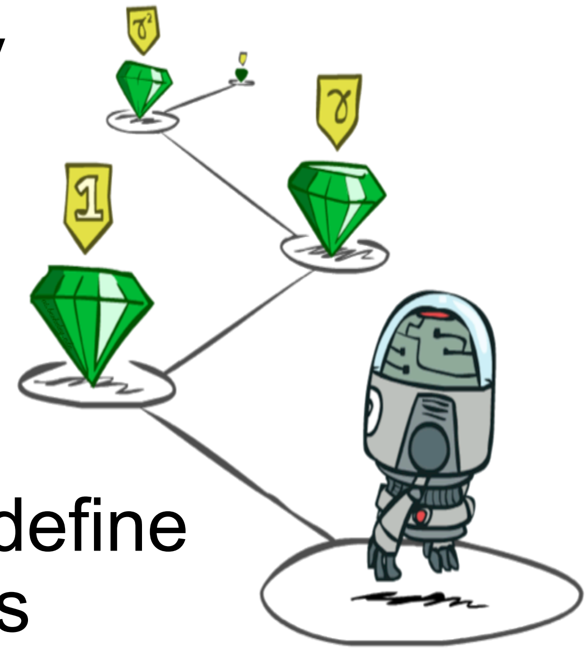
  [1, 2, 2] or [2, 3, 4]

- Now or later?

  [0, 0, 1] or [1, 0, 0]



Stationary Preferences



- Theorem: if we assume stationary preferences:

$$[a_1, a_2, \ldots] \succ [b_1, b_2, \ldots]$$
$$\Updownarrow$$
$$[r, a_1, a_2, \ldots] \succ [r, b_1, b_2, \ldots]$$

- Then: there are only two ways to define utilities over sequences of rewards
  - Additive utility: $U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots$
  - Discounted utility: $U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$

What are Discounts?

- It's reasonable to prefer rewards now to rewards later
- Decay rewards exponentially

1

Worth Now

γ

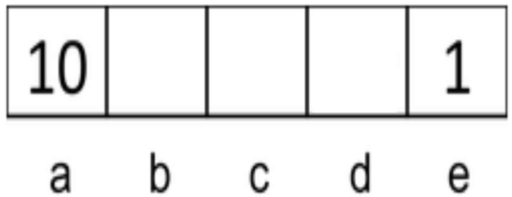
Worth Next Step

γ^2

Worth In Two Steps

Discounting $U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$

- Given:



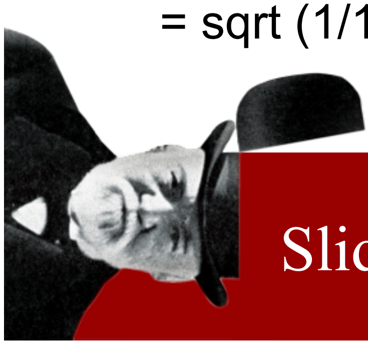
  - Actions: East, West
  - Terminal states: a and e (end when reach one or the other)
  - Transitions: deterministic
  - Reward for reaching a is 10 (regardless of initial state & action, e.g. r(s,action,a) = 10), reward for reaching e is 1, and the reward for reaching all other states is 0
- Quiz 1: For γ = 1, what is the optimal policy?
- Quiz 2: For γ = 0.1, what is the optimal policy for states b, c and d?
- Quiz 3: For which γ are West and East equally good when in state d?

Quiz: Discounting

- Given:

10				1
a	b	c	d	e

  - Actions: East, West
  - Terminal states: a and e (endwhen reach one or the other)
  - Transitions: deterministic
  - Reward for reaching a is 10 (regardless of initial state a& action, e.g. r(s,action,a) = 10), reward for reaching e is 1, and the reward for reaching all other states is 0
- Quiz 1: For γ = 1, what is the optimal policy?
  - In all states, Go West (towards a)
- Quiz 2: For γ = 0.1, what is the optimal policy?
  - b=W, c=W, d=E
- Quiz 3: For which γ are West and East equally good when in state d? Gamma = sqrt (1/10)

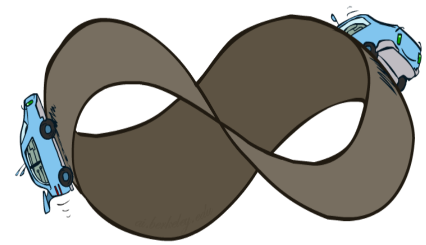


INFINITE UTILITIES?!

- Problem: What if the game lasts forever? Do we get infinite rewards?
- Solutions:
  - Finite horizon: (similar to depth-limited search)
    - Terminate episodes after a fixed T steps (e.g. li
    - Gives nonstationary policies (π depends on time left)
  - Discounting: use $0 < \gamma < 1$

$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1-\gamma)$$

    - Smaller γ means smaller "horizon" – shorter term focus
  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like "overheated" for racing)



Recap: Defining MDPs

- Markov decision processes:
  - Set of states S
  - Start state s_0
  - Set of actions A
  - Transitions P(s'|s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount γ)
- MDP quantities so far:
  - Policy = Choice* of action for each state
  - Utility/Value = sum of (discounted) rewards

Value of a Policy in Each State

- Expected immediate reward for taking action prescribed by policy π for that state
- And expected future reward get after taking that action from that state and following π

$$V^{\pi}(s) = \sum_{s' \in S} p(s' \mid s, \pi(s)) \left[R(s, \pi(s), s') + \gamma V^{\pi}(s') \right]$$

- Future reward depends on horizon (how many more steps get to act). For now assume infinite

Q: State-Action Value

- Expected immediate reward for taking action
- And expected future reward get after taking that action from that state and following π

$$Q^{\pi}(s,a) = \sum_{s' \in S} p(s' \mid s,a)\left[R(s,a,s') + \gamma V^{\pi}(s')\right]$$

Optimal Value V* and π*

- Optimal value: Highest possible value for each s
- Satisfies the **Bellman Equation**

$$V^*(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s_i, a, s') + \gamma V^*(s_j) \right] \right)$$

- Optimal policy

$$\pi^*(s_i) = \operatorname*{argmax}_a Q(s_i, a)$$

$$= \operatorname*{argmax}_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s_i, a, s') + \gamma V^*(s_j) \right] \right)$$

- Want to find these optimal values!

Value Iteration

- Bellman equation inspires an update rule

$$V^*(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s, a, s') + \gamma V^*(s_j) \right] \right)$$

$$\Downarrow$$

$$V_k(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s, a, s') + \gamma V_{k-1}(s_j) \right] \right)$$

- Form of *dynamic programming*

Also Called a Bellman Backup

$$V_k(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s,a,s') + \gamma V_{k-1}(s_j) \right] \right)$$

- In shorthand, for performing the above computation for all states,

$$V_k = BV_{k-1}$$

Carnegie Mellon University

Value Iteration Algorithm

1. Initialize $V_0(s_i)$=0 for all states $s_{i,}$ Set K=1
2. While k < desired horizon or (if infinite horizon) values have converged
   - For all s,

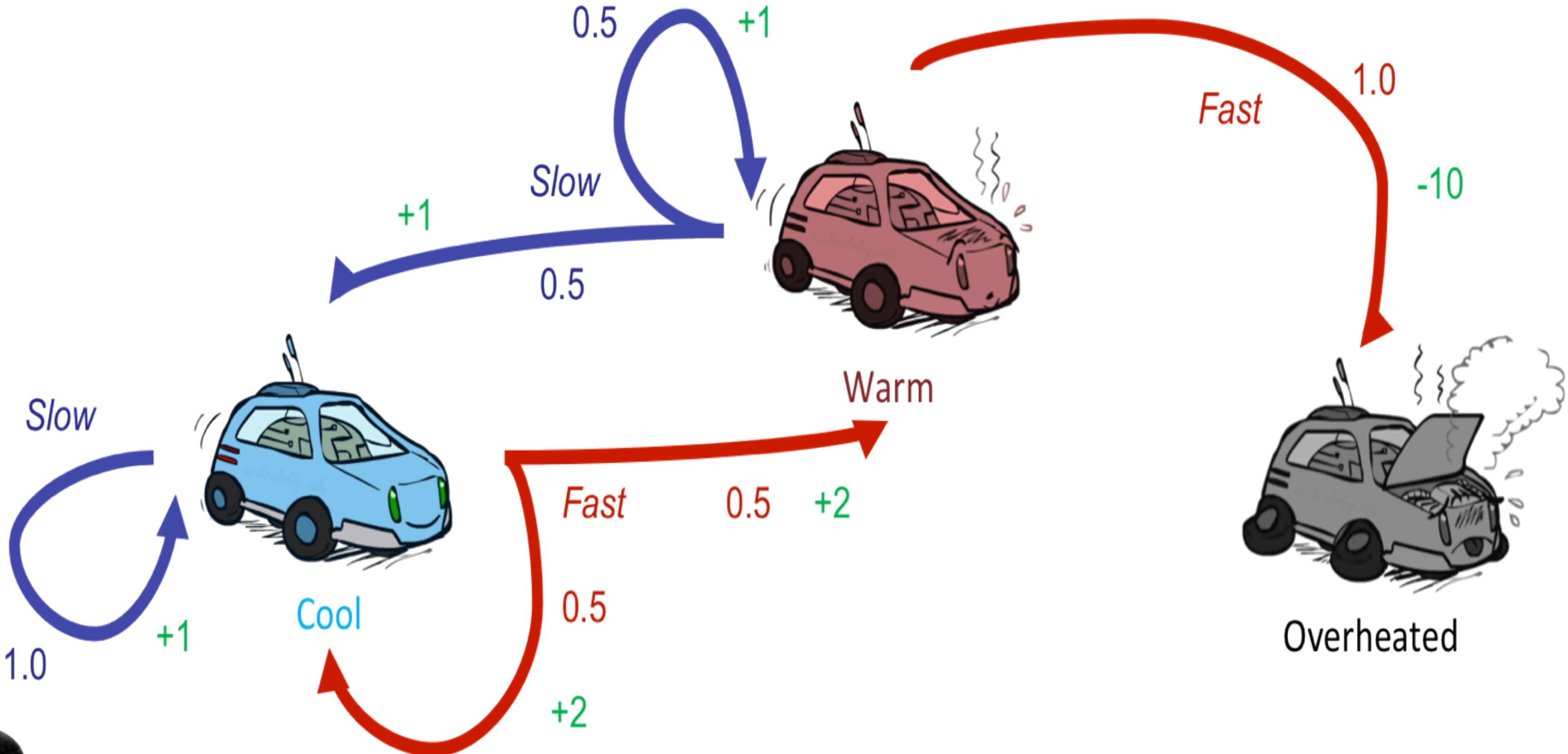
$$V_k(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s,a,s') + \gamma V_{k-1}(s_j) \right] \right)$$

3. Extract Policy

$$\pi_k(s_i) = \operatorname*{argmax}_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s,a,s') + \gamma V_{k-1}(s_j) \right] \right)$$

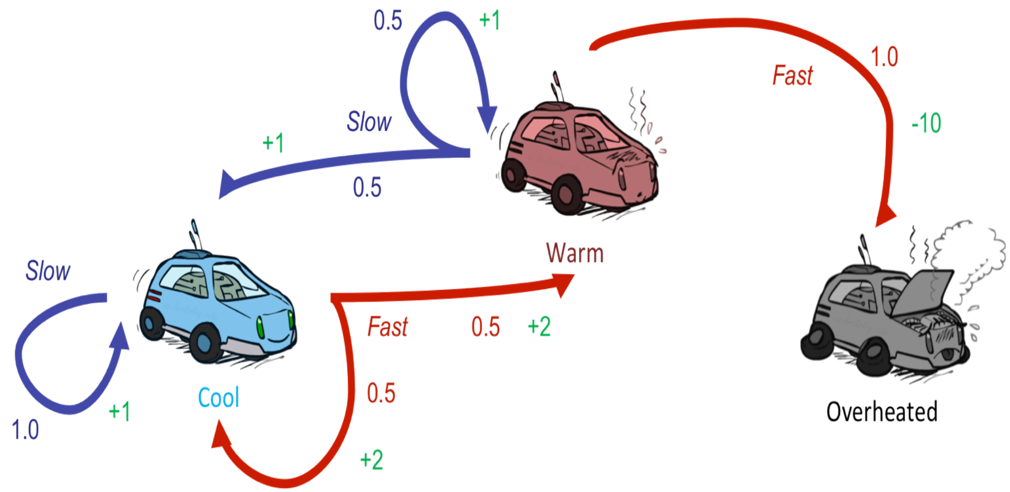
Calculate V_2(warmCar)

Assume Υ=1



For General Practice, check can calculate V_2(warmCar)

V_2			
V_1	2	1	0
V_0	0	0	0



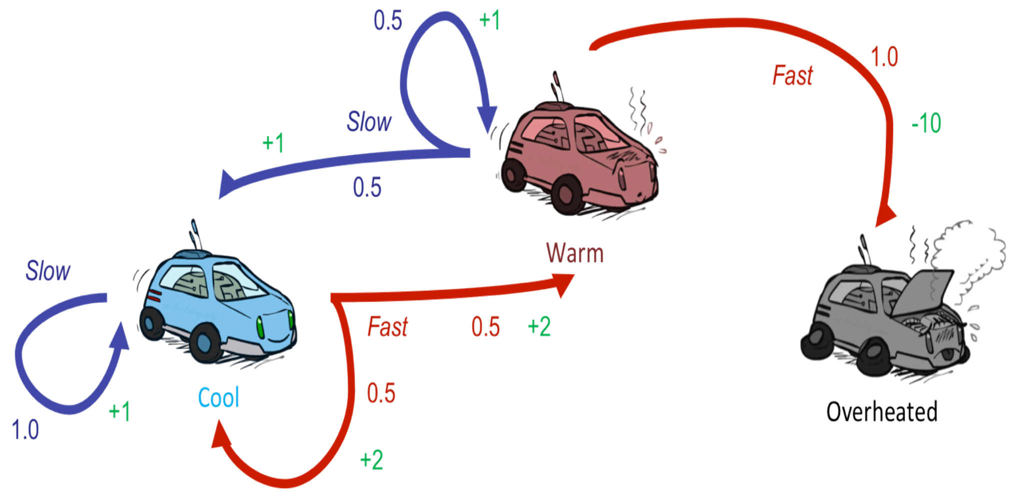
Assume ϒ=1

$$V_k(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s,a,s') + \gamma V_{k-1}(s_j) \right] \right)$$

Value Iteration

$V_2($  $)$: 2.5

V_2			
V_1	2	1	0
V_0	0	0	0



Assume ϒ=1

$$V_k(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s,a,s') + \gamma V_{k-1}(s_j) \right] \right)$$

Computational Cost for 1 Update of V(s) for all s in Value Iteration?

- For all s,

$$V_k(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s,a,s') + \gamma V_{k-1}(s_j) \right] \right)$$

Computational Cost Per Iteration?

AS^2

- For all s,

$$V_k(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s, a, s') + \gamma V_{k-1}(s_j) \right] \right)$$

Will Value Iteration Converge for Infinite Horizon Problems?

Contraction Operator

- Let O be an operator
- If |OV – OV’| <= |V-V|’
- Then O is a contraction operator

Will Value Iteration Converge?

- Yes, if discount factor $\gamma < 1$ or end up in a terminal state with probability 1
- Bellman equation is a contraction if discount factor, $\gamma < 1$
- If apply it to two different value functions, distance between value functions shrinks after apply Bellman equation to each

Bellman Operator is a Contraction (γ<1)

|| V-V'|| = Infinity norm (find max difference over all states, e.g. max(s) |V(s) – V'(s) |

$$\|BV - BV'\| = \left\| \max_a \left[R(s,a) + \gamma \sum_{s_j \in S} p(s_j \mid s_i, a) V(s_j) \right] - \max_{a'} \left[R(s,a') - \gamma \sum_{s_j \in S} p(s_j \mid s_i, a') V'(s_j) \right] \right\|$$

$$\leq \max_a \left\| \left[R(s,a) + \gamma \sum_{s_j \in S} p(s_j \mid s_i, a) V(s_j) - R(s,a) + \gamma \sum_{s_j \in S} p(s_j \mid s_i, a) V'(s_j) \right] \right\|$$

$$\leq \gamma \max_a \left\| \left[\sum_{s_j \in S} p(s_j \mid s_i, a) V(s_j) - \sum_{s_j \in S} p(s_j \mid s_i, a) V'(s_j) \right] \right\|$$

$$= \gamma \max_a \left\| \left[\sum_{s_j \in S} p(s_j \mid s_i, a)(V(s_j) - V'(s_j)) \right] \right\|$$

$$\leq \gamma \max_{a, s_i} \sum_{s_j \in S} p(s_j \mid s_i, a) \left| V(s_j) - V'(s_j) \right|$$

$$\leq \gamma \max_{a, s_i} \sum_{s_j \in S} p(s_j \mid s_i, a) \left\| V - V' \right\|$$

$$= \gamma \left\| V - V' \right\|$$

Properties of Contraction

- Only has 1 fixed point (the point reach if apply a contraction operator many times)
  - If had two, then would not get closer when apply contraction function, violating definition of contraction
- When apply contraction function to any argument, value must get closer to fixed point
  - Fixed point doesn't move
  - Repeated function applications yield fixed point

VI Converges

- Value iteration converges to unique solution which is optimal value function

- Proof: $\lim_{k\to\infty} V_k = V^*$

$$\|V_{k+1} - V^*\|_\infty = \|BV_k - V^*\|_\infty \le \gamma \|V_k - V^*\|_\infty \le \ldots$$
$$\le \gamma^{k+1} \|V_0 - V^*\|_\infty \to 0$$

Discuss and Report Back: Does Initialization Impact Final Value?

Value Iteration Algorithm

1. Init $V_0(s_i)$ for all states s_i
2. k=1
3. While k < desired horizon or (if infinite horizon) values have converged
   - For all s,

$$V_k(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a)\left[R(s,a,s') + \gamma V_{k-1}(s_j)\right]\right)$$