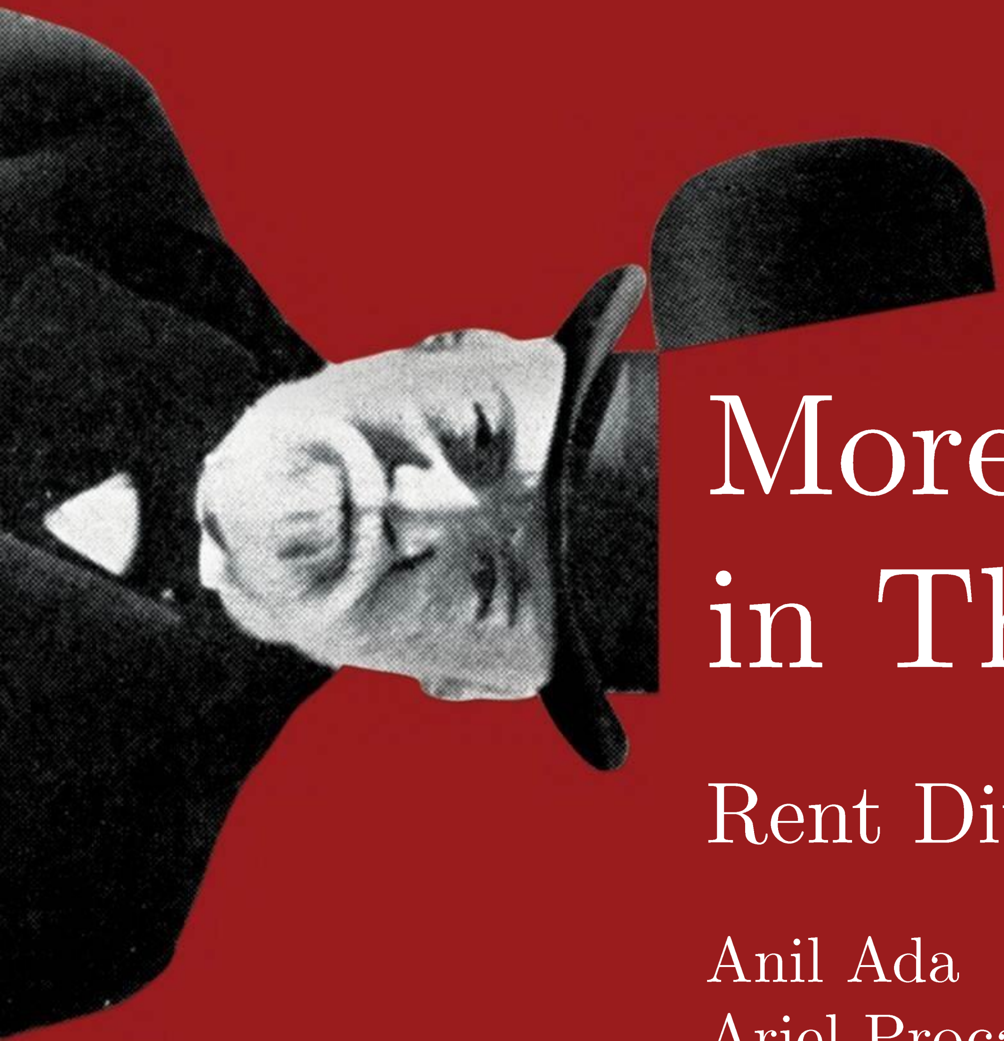


More Great Ideas in Theoretical CS

Rent Division

Anil Ada
Ariel Procaccia (this time)

The Whining Philosophers Problem

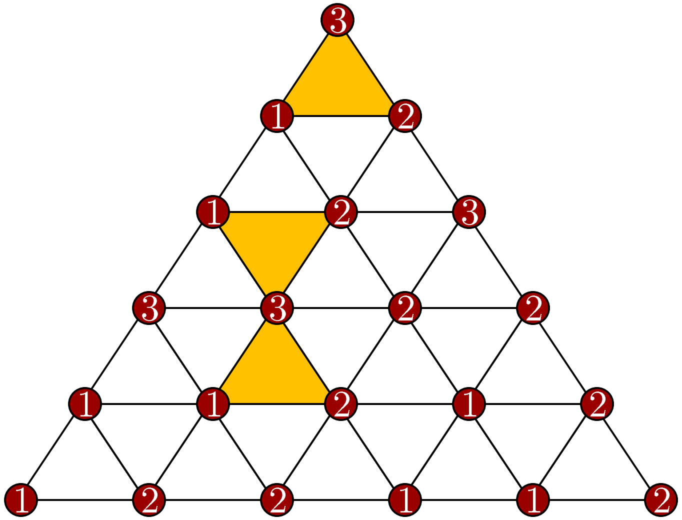
Nir Ben Moshe

Naomi Sender

Ariel Procaccia

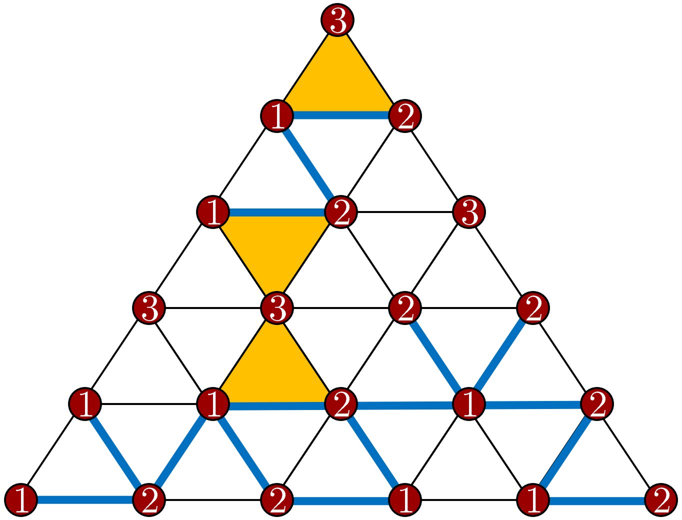
Sperner's Lemma

- Triangle T partitioned into elementary triangles
- Label vertices by {1,2,3} using Sperner labeling:
  - Main vertices are different
  - Label of vertex on an edge (i, j) of T is i or j
- Lemma: Any Sperner labeling contains at least one fully labeled elementary triangle



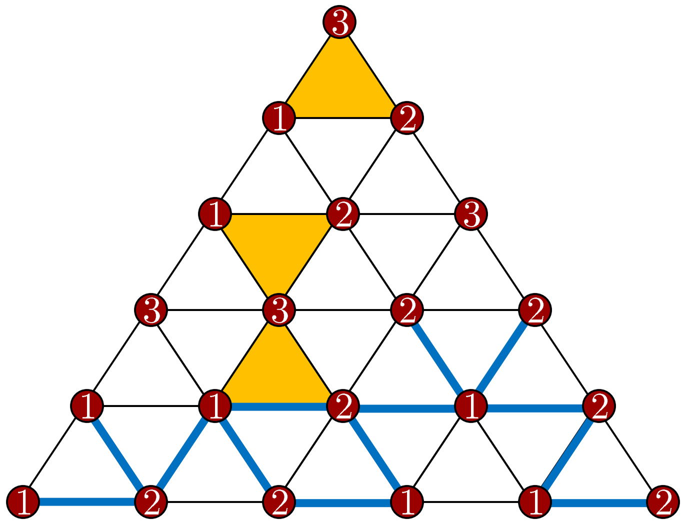
Proof of Lemma

- Doors are 12 edges
- Rooms are elementary triangles
- #doors on the boundary of T is odd
- Every room has ≤ 2 doors; one door iff the room is 123



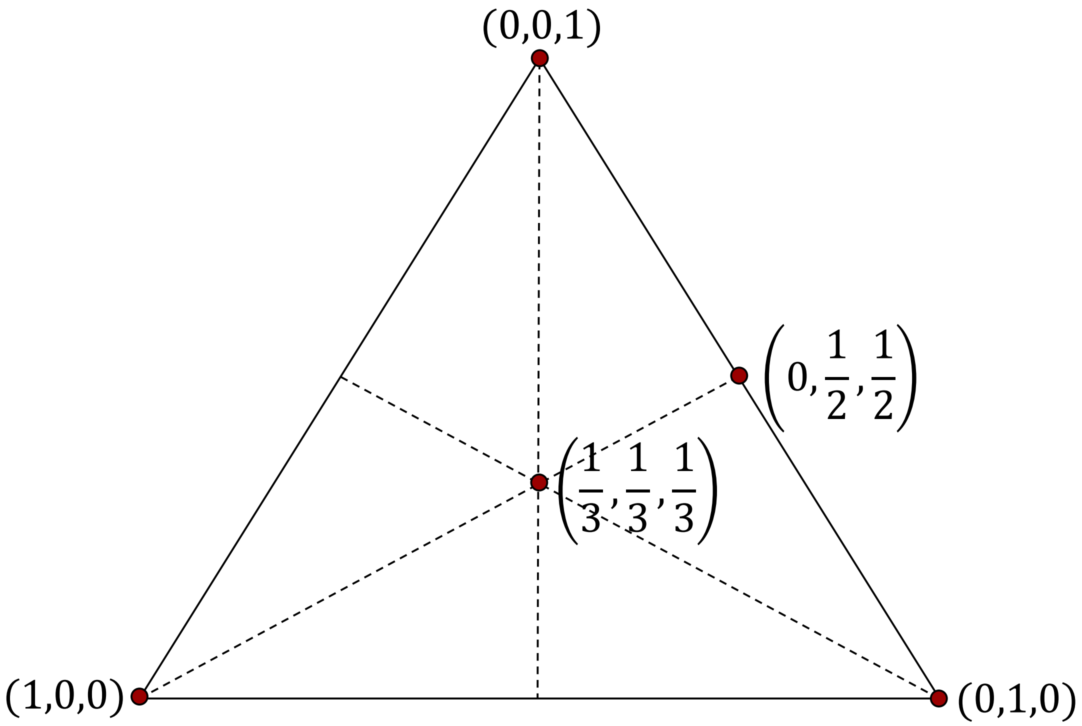
Proof of Lemma

- Start at door on boundary and walk through it
- Room is fully labeled or it has another door…
- No room visited twice
- Eventually walk into fully labeled room or back to boundary
- But #doors on boundary is odd ■



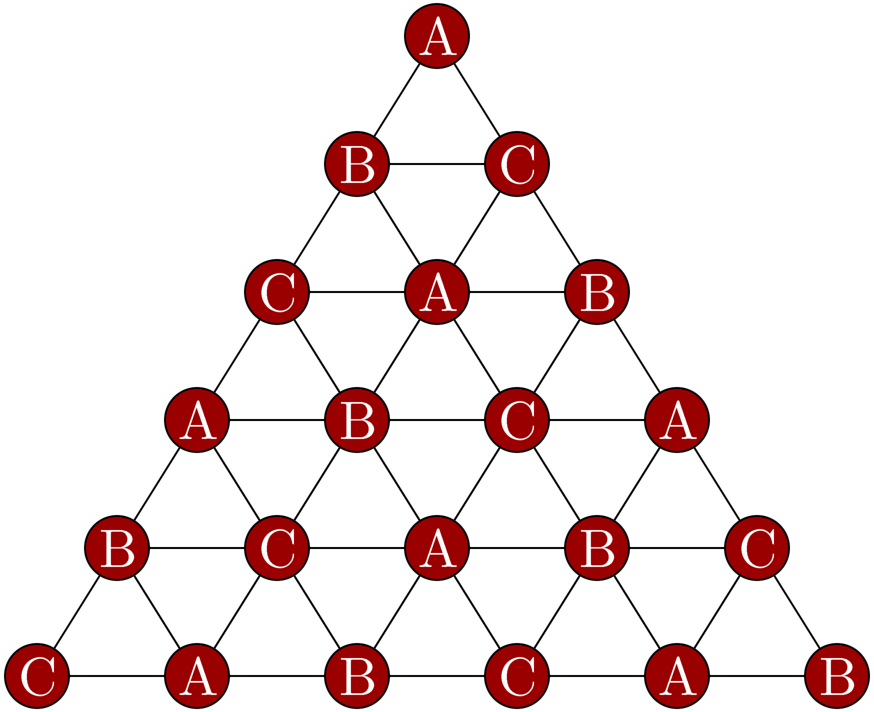
Fair rent division

- Assume there are three housemates A, B, C
- Goal is to divide rent so that each person wants different room
- Sum of prices for three rooms is 1
- Can represent possible partitions as triangle


(0,0,1)
$\left(0, \frac{1}{2}, \frac{1}{2}\right)$
$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
(1,0,0)
(0,1,0)

Fair rent division

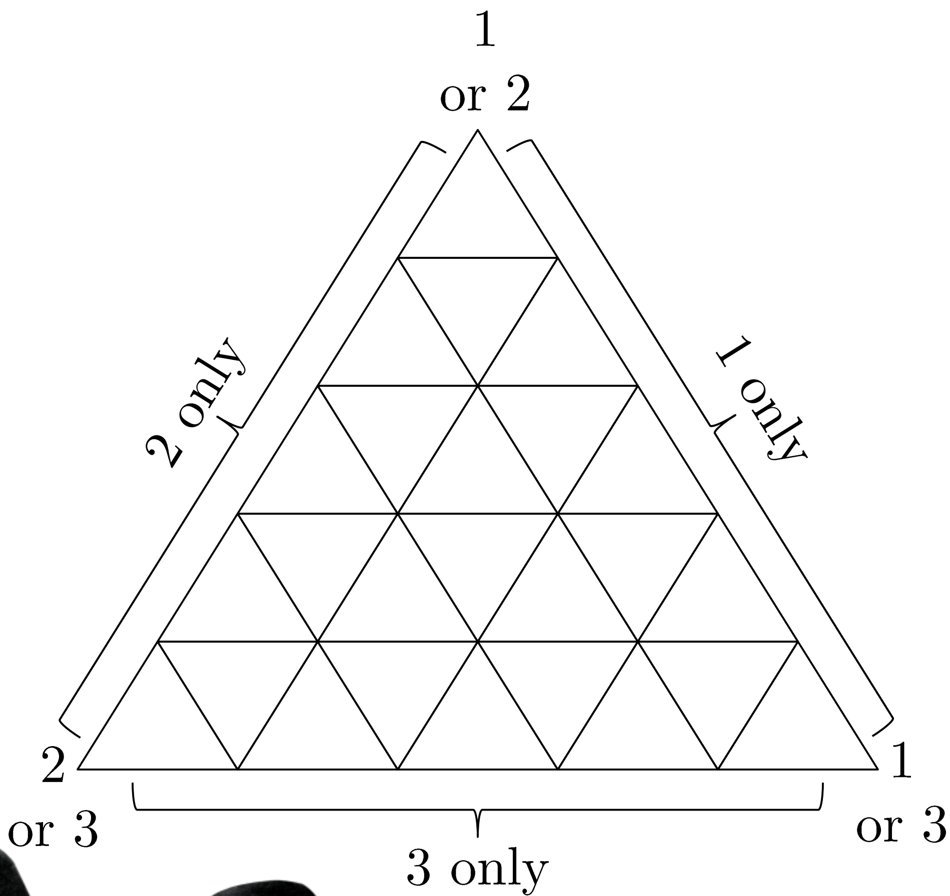
- “Triangulate” and assign “ownership” of each vertex to each of A, B, and C …
- … in a way that each elementary triangle is an ABC triangle

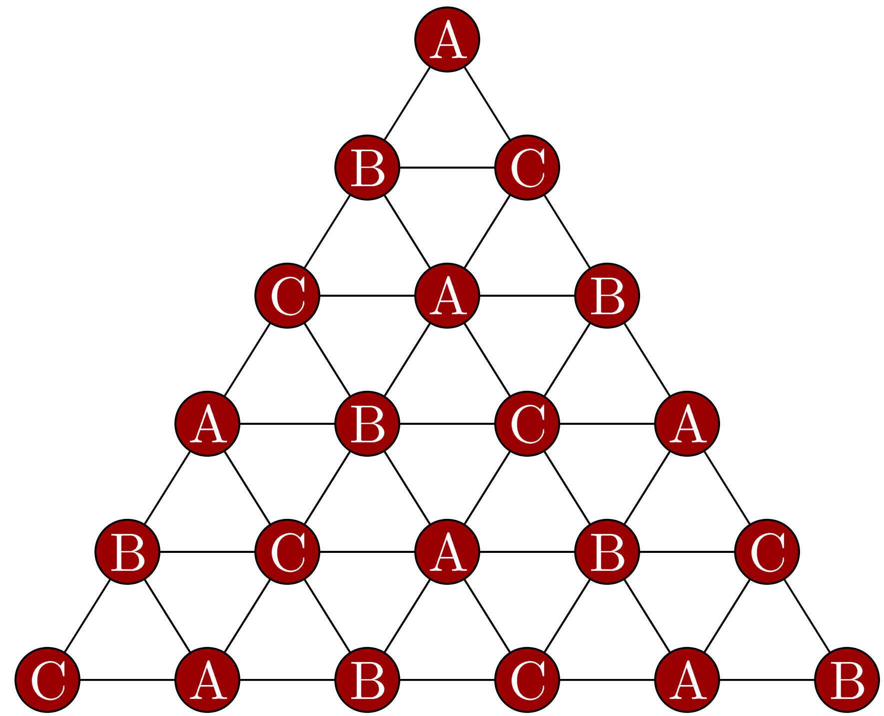

A
B C
C A B
A B C A
B C A B C
C A B C A B

Fair rent division

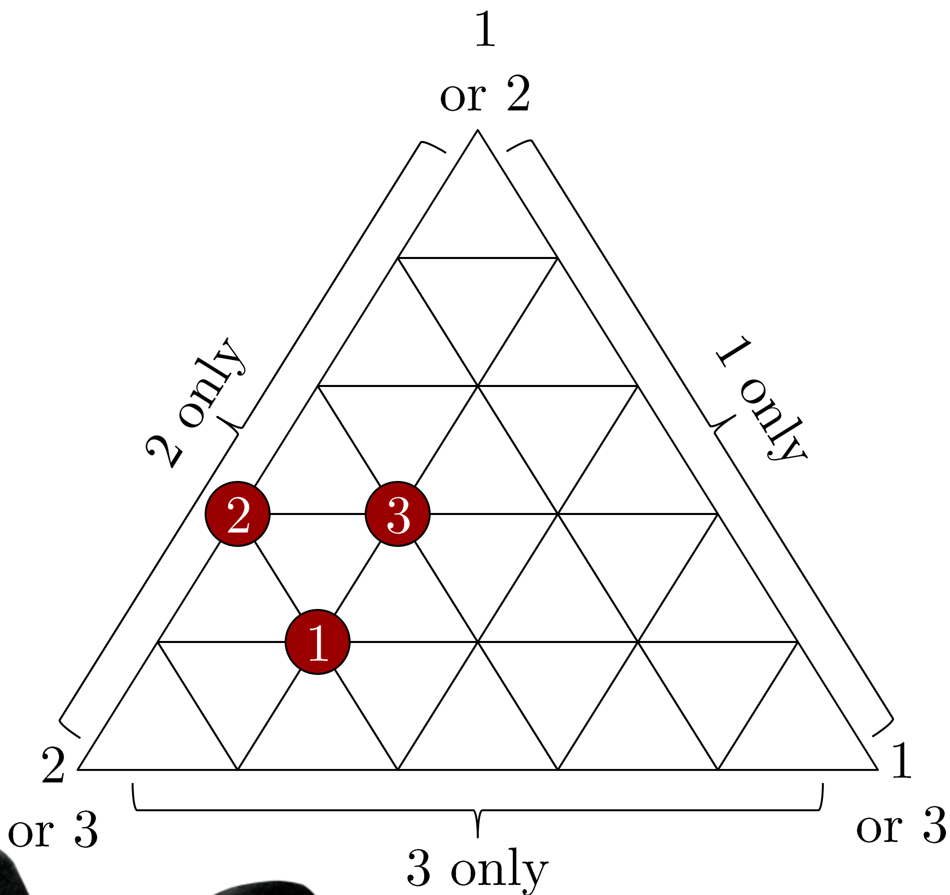
- Ask the owner of each vertex to tell us which room he prefers
- This gives a new labeling by 1, 2, 3
- Assume that a person wants a free room if one is offered to him

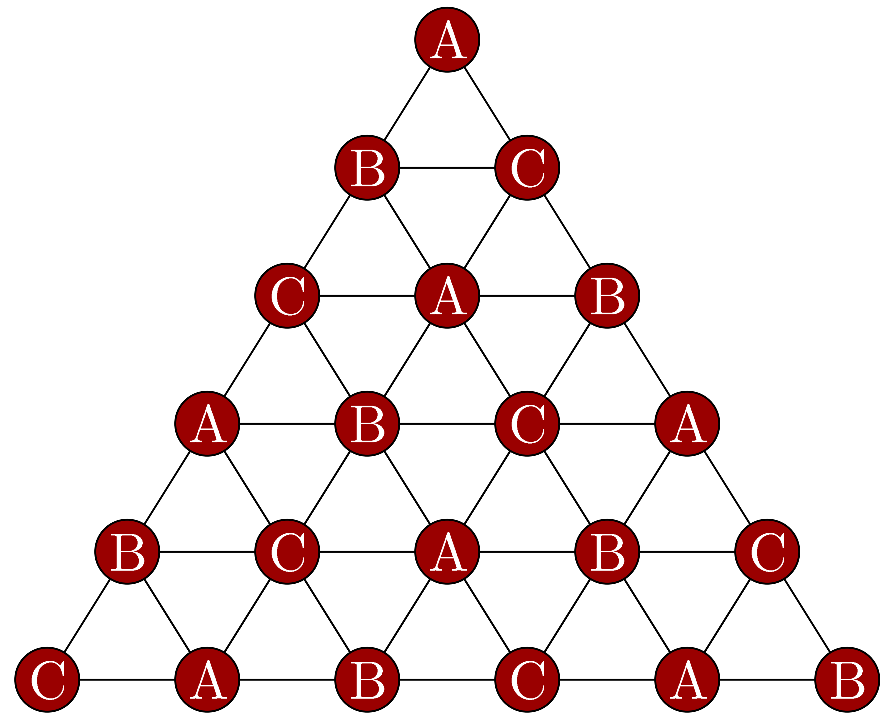
- Choice of rooms on edges is constrained by the free room assumption





- Sperner's lemma (variant): such a labeling must have a 123 triangle





Fair rent division

- Such a triangle is nothing but an approximately envy free allocation!
- By making the triangulation finer, we can increase accuracy
- In the limit we obtain a completely envy free allocation
- Same techniques generalize to more housemates [Su 1999]

Quasi-linear utilities

- Suppose each player $i \in N$ has value v_{ir} for room r
- $\sum_r v_{ir} = 1$, where 1 is the total rent
- The utility of player i for getting room r at price p_r is $v_{ir} - p_r$
- $(\pi, \boldsymbol{p})$ is envy free if

$$\forall i, j \in N, v_{i\pi(i)} - p_{\pi(i)} \geq v_{i\pi(j)} - p_{\pi(j)}$$

- Theorem [Svensson 1983]: An envy-free solution always exists under quasi-linearity

Which model is better?

- Advantage of quasi-linear utilities: preference elicitation is easy
  - Each player reports a single number in one shot
- Disadvantage of quasi-linear utilities: does not correctly model real-world situations
  - I want the room but I really can't spend more than $500 on rent