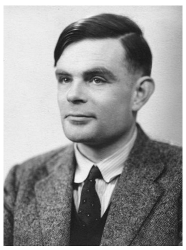
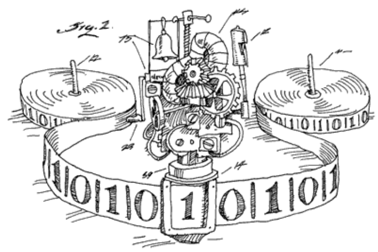
15-251
Great Ideas in
Theoretical Computer Science

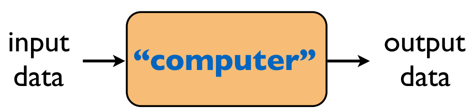
Lecture 5:
Turing's Legacy: Turing Machines





September 12th, 2017

This Week



What is computation?

What is an algorithm?

How can we mathematically define them?

Goal of this lecture:

Define Turing machines.

Understand how they work.

Goal of next lecture:

Explore physical, philosophical, historical questions surrounding Turing machines.

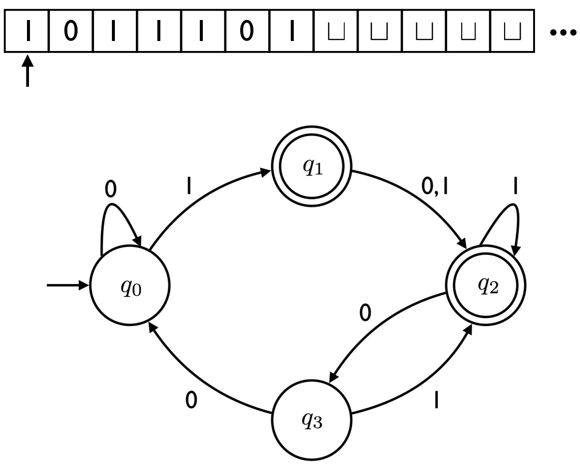
Let’s assume two things about our world

1. No “universal” machines exist.

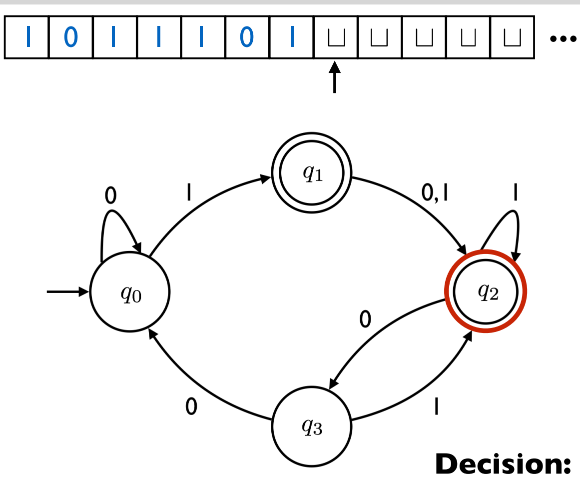
+ | **isPrime** | **Sorting** | **DFA |x| even?**

2. We only have machines to solve decision problems.

DFA: state diagram + input tape



DFA: state diagram + input tape



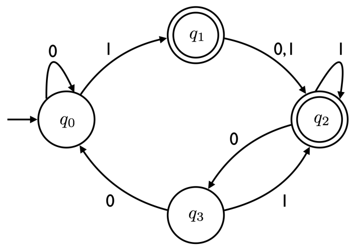
Decision: Accept

DFA as a programming language

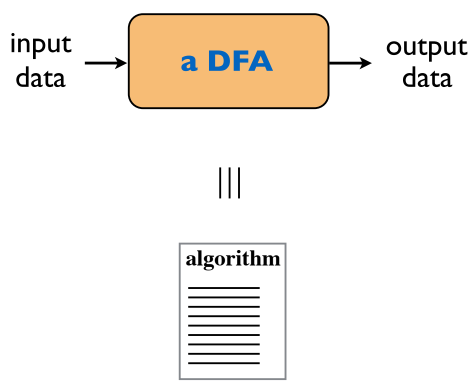
input = | 0 | 1 | 1 | 1 | 1 |

```
def foo(input):
  i = 0;
  STATE 0:
    if (i == input.length): return False;
    letter = input[i];
    i++;
    switch(letter):
      case '0':  go to STATE 0;
      case '1':  go to STATE 1;

  STATE 1:
    if (i == input.length): return True;
    letter = input[i];
    i++;
    switch(letter):
      case '0':  go to STATE 2;
      case '1':  go to STATE 2;
  ...
```



machine ≈ algorithm describing it

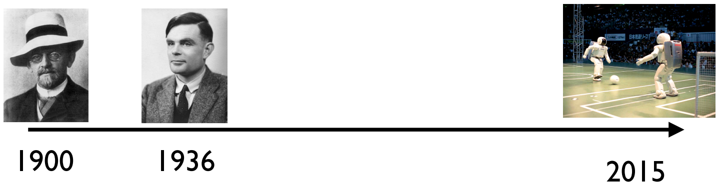


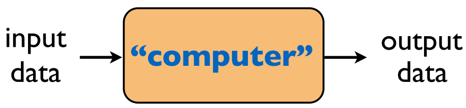
What is computation?

What is an algorithm?

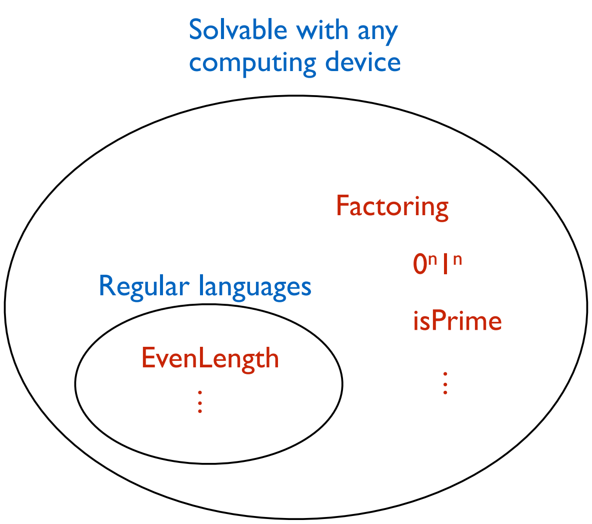
How can we mathematically define them?

The properties we want from the definition:





2 important observations:



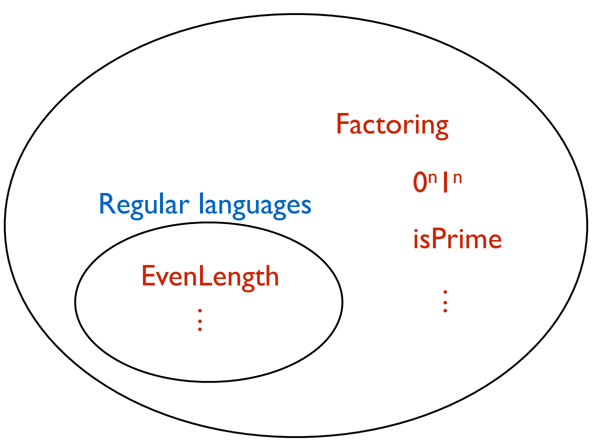
Solving 0^n1^n in Python

```
def foo(input):
    i = 0
    j = len(input) - 1
    while(j >= i):
        if(input[i] != ‘0’ or input[j] != ‘1’):
            return False
        i = i + 1
        j = j - 1
    return True
```

Solving 0^n1^n in C

```
int foo(char input[])
{
    int i = 0, j;
    while(input[j] != NULL)  /* NULL is end-of-string character */
        j++;
    j—;
    while(j >= i)
    {
        if(input[i] != ‘0’ || input[j] != ‘1’)
            return 0;  /* Reject */
        i++;
        j—;
    }
    return 1;  /* Accept */
}
```

Solvable with **Python**???



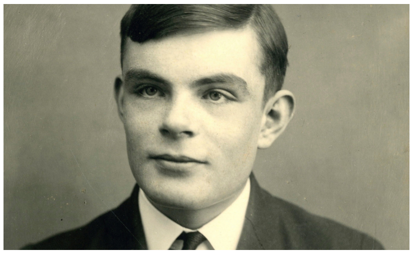
Should we define **computable** to mean
what is computable by a Python function/program?

Downsides as a formal definition?

So what we want is:

A totally minimal (TM) programming language such that:

Actually TM™ stands for Turing machine.



Defined by Alan Turing in a paper he wrote in 1936
while he was a PhD student.

Turing machine description

TM ≈ DFA + infinite tape

-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
⊔	⊔	⊔	a	a	b	a	⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔

Turing machine description

TM ≈ DFA + infinite tape

-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
⊔	⊔	⊔	a	a	b	a	⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔

TM could have been defined as a sequence of instructions, where the allowed instructions are:

> Move the head left
> Move the head right
> Write a symbol a (from the alphabet)
> If head is reading symbol a, GOTO step j
> Halt and accept
> Halt and reject

But, we want to keep the definition as simple as possible.

Turing machine description

TM ≈ DFA + infinite tape

-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
⊔	⊔	⊔	a	a	b	a	⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔

So a TM is a sequence of steps (states), each looking like:

STATE 0:
  switch(letter under the head):
    case ‘a’: **write** ‘b’; **move** Left; **go to** STATE 2;
    case ‘b’: **write** ‘⊔’; **move** Right; **go to** STATE 0;
    case ‘⊔’: **write** ‘b’; **move** Left; **go to** STATE 1;

Turing machine description

STATE 0:
  switch(letter under the head):
    case 'a': **write** 'b'; **move** Left; **go to** STATE 2;
    case 'b': **write** '⊔'; **move** Right; **go to** STATE 0;
    case '⊔': **write** 'b'; **move** Left; **go to** STATE 1;

At each step, you have to:

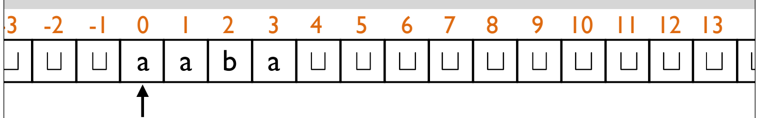
- write a *new* symbol to the cell under the head
- move tape head Left or Right
- go to a *new* state

Don't want to change the symbol:

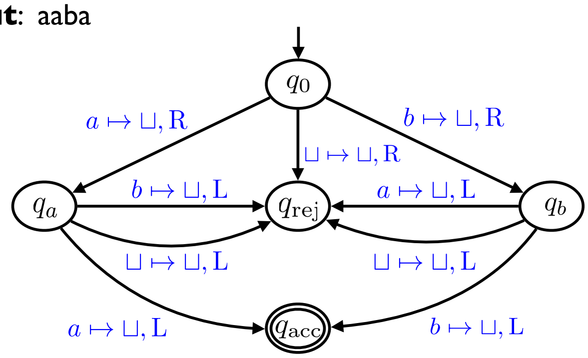
Want to stay put:

Don't want to change state:

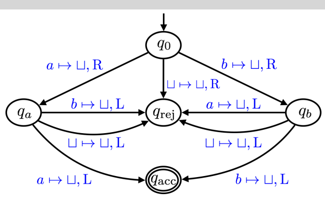
Turing machine official picture



Input: aaba



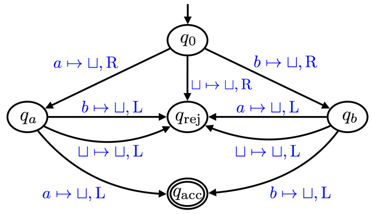
TM as a programming language




```
def M(input):
   i = 0
   STATE 0:
      letter = input[i];
      switch(letter):
         case 'a': input[i] = ' '; i++; go to STATE a;
         case 'b': input[i] = ' '; i++; go to STATE b;
         case ' ': input[i] = ' '; i++; go to STATE rej;
   STATE a:
      letter = input[i];
      switch(letter):
         case 'a': input[i] = ' '; i--; go to STATE acc;
         case 'b': input[i] = ' '; i--; go to STATE rej;
         case ' ': input[i] = ' '; i--; go to STATE rej;

   ⋮
```

Poll



The machine accepts a string x if and only if:

Exercise

Let $\Sigma = \{a, b\}$.

Draw the state diagram of a TM that accepts a string iff it starts and ends with an a.

Formal definition: Turing machine

A Turing machine (TM) M is a 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{rej})$$

where

- Q is a finite set (which we call the set of states);
- Σ is a finite set with $\sqcup \notin \Sigma$ (which we call the input alphabet);
- Γ is a finite set with $\sqcup \in \Gamma$ and $\Sigma \subset \Gamma$ (which we call the tape alphabet);
- δ is a function of the form $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ (which we call the transition function);
- $q_0 \in Q$ (which we call the start state);
- $q_{\text{acc}} \in Q$ (which we call the accept state);
- $q_{\text{rej}} \in Q$, $q_{\text{rej}} \neq q_{\text{acc}}$ (which we call the reject state);

Formal definition: TM accepting a string

A bit more involved to define rigorously.

Not too much though.

See course notes.

DFAs vs TMs

Definition: decidable/computable languages

Let M be a Turing machine.

We let $L(M)$ denote the set of strings that M accepts.

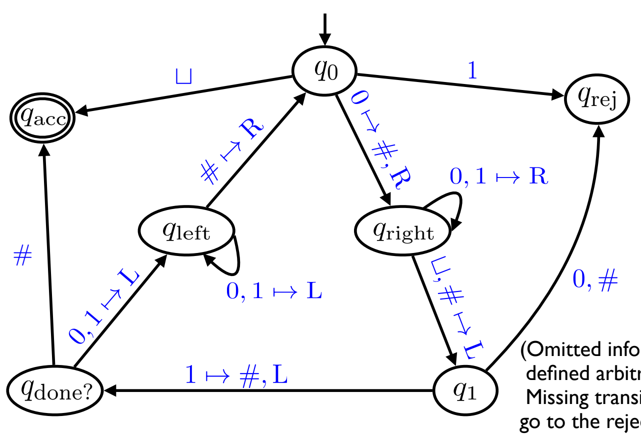
So, $L(M) = \{x \in \Sigma^* : M(x) \text{ accepts.}\}$

What is the analog of regular languages in this setting?

regular languages $\stackrel{?}{=}$ decidable languages

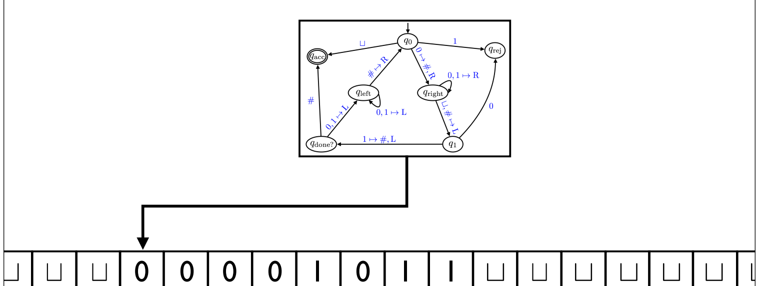
Turing machine that decides $0^n 1^n$

$\Sigma = \{0, 1\}$ $\qquad$ $\Gamma = \{0, 1, \#, \sqcup\}$



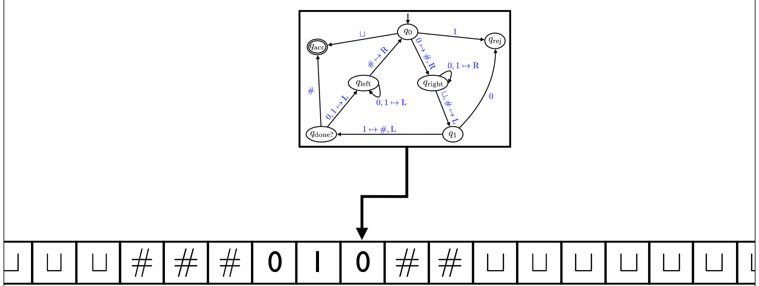
(Omitted information defined arbitrarily. Missing transitions go to the reject state.)

Turing machine that decides $0^n 1^n$



Input: 00001011

Turing machine that decides 0^n1^n



Input: 00001011 **Decision**: reject

Some TM subroutines and tricks

- Move right (or left) until first $\sqcup$ encountered
- Shift entire input string one cell to the right
- Convert input from

  $x_1x_2x_3\ldots x_n$ to $\sqcup x_1 \sqcup x_2 \sqcup x_3 \ldots \sqcup x_n$
- Simulate a big Γ by just $\{0, 1, \sqcup\}$
- "Mark" cells. If $\Gamma = \{0, 1, \sqcup\}$, extend it to

  $\Gamma = \{0, 1, 0^\bullet, 1^\bullet, \sqcup\}$
- Copy a stretch of tape between two marked cells into another marked section of the tape

Some TM subroutines and tricks

- Implement basic string and arithmetic operations
- Simulate a TM with 2 tapes and heads
- Implement basic data structures
- Simulate "random access memory"

  ⋮
- Simulate assembly language

  You could prove this rigorously if you wanted to.

So what we want is:

A totally minimal (TM) programming language such that

- it can simulate simple bytecode (and therefore Python, C, Java, SML, etc…) ✓
- it is simple to define and reason about completely mathematically rigorously ✓

A note

You could describe a TM in 3 ways:

Low level description

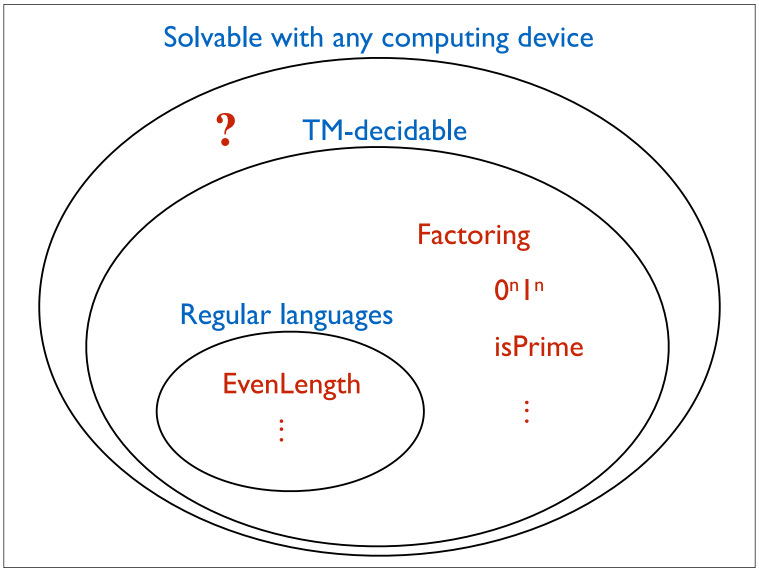
Medium level description

High level description

Important Question

Is TM the right definition?

Is there a reasonable definition of "algorithm" that can compute more languages than TM-decidable ones?


Solvable with any computing device
?
TM-decidable
Factoring
0^n1^n
isPrime
⋮
Regular languages
EvenLength
⋮